Electronic Journal of University of Aden for Basic and Applied Sciences
EJUA-BA Vol. 1 No. 2 (2020)
https://ejua.net/index.php/ejua-ba/article/view/19
ISSN: 2708-0684

RESEARCH ARTICLE

A NEW EXTENSIONS OF GEGENBAUER POLYNOMIALS

Ahmed Ali Atash[1,*] and Ahmed Ali Al-Gonah[2]

[1]*Department of Mathematics, Faculty of Education-Shabwah, Aden University, Shabwah, Yemen*
[2]*Department of Mathematics, Faculty of Science, Aden University, Yemen*

***Corresponding author: Ahmed Ali Atash; E-mail: ah-a-atash@hotmail.com**

Received: 07 April 2020 / Accepted: 10 June 2020 / Published online: 30 June 2020

Abstract

The main aim of this paper is to introduce new extensions of Gegenbauer polynomials of one and two variables by using the extended Gamma function given by Chaudhry and Zubair [3]. Some properties of these extended polynomials such as generating functions, integral representations, and Mellin transform are deduced.

Keywords: Gamma function, Extended Gegenbauer polynomials, Generating functions integral representations, Mellin transform.

2010 Mathematics Subject Classification - 33B15, 33B20, 33C05, 33C20, 33C45, 33C60.

1. Introduction

In 1994, Chaudhry and Zubair [3] have introduced the following extension of the Gamma function:

$$\Gamma_{ap}(\alpha) = a^{\alpha}\int_0^{\infty} t^{\alpha-1}\exp(-at-pt^{-1})dt, \tag{1.1}$$

$a+b>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(\alpha)>0$.

Clearly, for $a=1$ (1.1) reduces to the generalized gamma function $\Gamma_p(\alpha)$ [3]

$$\Gamma_p(\alpha) = \begin{cases} \int_0^{\infty} t^{\alpha-1}\exp(-t-pt^{-1})dt & \mathrm{Re}(p)>0, \\ \Gamma(\alpha) & (p=0,\ \mathrm{Re}(\alpha)>0), \end{cases} \tag{1.2}$$

where $\Gamma(\alpha)$ is the well-known gamma function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\exp(-t)dt,\ \ \mathrm{Re}(\alpha)>0. \tag{1.3}$$

The generalized Pochhammer symbol is defined by (see [12])

$$(\lambda;p)_{\nu} = \begin{cases} \dfrac{\Gamma_{(p)}(\lambda+\nu)}{\Gamma(\lambda)}, & \mathrm{Re}(p)>0, \lambda,\nu\in C \\ (\lambda)_{\nu}, & p=0, \lambda,\nu\in C \end{cases} \tag{1.4}$$

where $(\lambda)_n$ is the familiar Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)(\lambda+2)....(\lambda+n-1) & (n\in N) \end{cases}. \tag{1.5}$$

In terms of the generalized Pochhammer symbol (1.4), Srivastava et al. [12] introduced the following generalized hypergeometric function

$${}_rF_s\left[\begin{matrix}(a_1;p), a_2, ..., a_r\ ; \\ b_1, ..., b_s\ ;\end{matrix}\ x\right] = \sum_{n=0}^{\infty}\frac{(a_1;p)_n (a_2)_n .... (a_r)_n}{(b_1)_n .... (b_s)_n}\frac{x^n}{n!}, \tag{1.6}$$

when $p=0$ (1.6) yields the generalized hypergeometric function ${}_rF_s()$ (see, [13])

$${}_rF_s\left[\begin{matrix}a_1, ..., a_r\ ; \\ b_1, ..., b_s\ ;\end{matrix}\ x\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n .... (a_r)_n}{(b_1)_n .... (b_s)_n}\frac{x^n}{n!}. \tag{1.7}$$

In this paper, we introduce and study the extended Gegenbauer polynomials of one and two variables by means of the extended Gamma function $\Gamma_{ap}(\alpha)$ and

$\Gamma_p(\alpha)$. For this aim, we recall that the two variable Gegenbauer polynomials is defined by the following series definition and integral representations (see [4]):

$$C_n^{\alpha}(x,y,a) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-y)^k\Gamma(\alpha+n-k)}{(n-2k)!k!\,a^{\alpha+n-k}} \qquad (1.8)$$

and

$$C_n^{\alpha}(x,y,a) = \frac{1}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha-1}\exp(-at)H_n(2xt,-yt)dt\ , \qquad (1.9)$$

where $H_n(x,y)$ denotes the two variables Hermit–Kamp de Friet polynomials defined by [1]

$$H_n(x,y) = n!\sum_{k=o}^{[\frac{n}{2}]}\frac{x^{n-2k}y^k}{(n-2k)!k!} \qquad (1.10)$$

and specified by the following generating function:

$$\sum_{n=0}^{\infty}\frac{H_n(x,y)t^n}{n!} = \exp(xt+yt^2)\ . \qquad (1.11)$$

Note that, for $a=1$ in (1.8), we obtain

$$C_n^{\alpha}(x,y) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-y)^k\Gamma(\alpha+n-k)}{(n-2k)!k!}\ . \qquad (1.12)$$

Also, for $y=a=1$ in (1.8), we obtain the classical Gegenbauer polynomials $C_n^{\alpha}(x)$ [9]

$$C_n^{\alpha}(x) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-1)^k\Gamma(\alpha+n-k)}{(n-2k)!k!}\ . \qquad (1.13)$$

2. An extended Gegenbauer polynomials of two variable

In terms of the extended Gamma function given in (1.1), we introduce a new extension of Gegenbauer polynomials $C_n^{\alpha}(x,y;ap)$ as follows:

$$C_n^{\alpha}(x,y;ap) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-y)^k\Gamma_{ap}(\alpha+n-k)}{(n-2k)!\,k!}\ . \qquad (2.1)$$

Clearly, when $a=1$ then we obtain a new extension of Gegenbauer polynomials $C_n^{\alpha}(x,y;p)$ in terms of the extended Gamma function given in (1.2) as follows:

$$C_n^{\alpha}(x,y;p) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-y)^k\Gamma_p(\alpha+n-k)}{(n-2k)!\,k!}\ . \qquad (2.2)$$

Note that, for $p=0$ in (2.2), we obtain the two variable Gegenbauer polynomial $C_n^{\alpha}(x,y)$ given in (1.12)

Remark 2.1. Taking y = 1 in (2.1) and (2.2), we get the following new extensions of Gegenbauer polynomials $C_n^{\alpha}(x;ap)$ and $C_n^{\alpha}(x;p)$ as follows:

$$C_n^{\alpha}(x;ap) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-1)^k\Gamma_{ap}(\alpha+n-k)}{(n-2k)!k!} \qquad (2.3)$$

and

$$C_n^{\alpha}(x;p) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-1)^k\Gamma_p(\alpha+n-k)}{(n-2k)!k!}\ . \qquad (2.4)$$

Note that, for $p=0$ in (2.4), we obtain the classical Gegenbauer polynomials $C_n^{\alpha}(x)$ given in (1.13).

Some properties of the above extended Gegenbauer polynomials of the two variables $C_n^{\alpha}(x,y;ap)$ and $C_n^{\alpha}(x,y;p)$ are established in the following Theorems:

Theorem 2.1 For $a>0$ the following integral representation for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;ap)$ holds true:

$$C_n^{\alpha}(x,y;ap) = \frac{a^{\alpha}}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha-1}\exp(-at-pt^{-1})H_n(2axt,-ayt)dt \qquad (2.5)$$

Proof. Consider the L.H.S. of equation (2.5) and using (2.1) and (1.1), we get

$$C_n^{\alpha}(x,y;ap) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{[\frac{n}{2}]}\frac{(2x)^{n-2k}(-y)^k\,a^{\alpha+n-k}}{(n-2k)!k}$$

$$\times\int_0^{\infty} t^{\alpha+n-k-1}\exp(-at-pt^{-1})dt$$

Now, interchanging the order of summation and integration and using (1.10), we obtain

$$C_n^{\alpha}(x,y;ap) = \frac{a^{\alpha}}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha-1}\exp(-at-pt^{-1})\, t^n\, H_n(2ax,-ay/t)\,dt$$

Finally, by using the following relation (see, [5]):

$$t^n H_n(x,y) = H_n(xt, yt^2) \;,$$

we obtain the desired result.

Remark 2.2. Replacing x and y by $\frac{x}{a}$ and $\frac{y}{a}$ in (2.5), we have the following result:

$$C_n^{\alpha}(x/a, y/a; ap)$$

$$= \frac{a^{\alpha}}{\Gamma(\alpha)\,n!}\int_0^{\infty} t^{\alpha-1}\exp(-at-pt^{-1})H_n(2xt,-yt)dt \;. \quad (2.6)$$

Remark 2.3. Taking $a=1$, $y=1$ and $y=a=1$ in (2.5), we get respectively the following results:

Corollary 2.1. The following integral representations of $C_n^{\alpha}(x,y;p)$, $C_n^{\alpha}(x;ap)$ and $C_n^{\alpha}(x;p)$ holds true:

$$C_n^{\alpha}(x,y;p) = \frac{1}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha-1}\exp(-t-pt^{-1})H_n(2xt,-yt)dt \;, \quad (2.7)$$

$$C_n^{\alpha}(x;ap)$$

$$= \frac{a^{\alpha}}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha-1}\exp(-at-pt^{-1})H_n(2axt,-at)dt \quad (2.8)$$

and

$$C_n^{\alpha}(x;p) = \frac{1}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha-1}\exp(-t-pt^{-1})H_n(2xt,-t)dt \;. \quad (2.9)$$

Theorem 2.2. The following Mellin transform representation of the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;ap)$ holds true:

$$\int_0^{\infty} C_n^{\alpha}(x,y;ap)\,p^{s-1}dp$$

$$= \frac{a^{\alpha}\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)} C_n^{\alpha+s}(ax,ay,a) \quad (2.10)$$

Proof. Multiplying both sides of equation (2.1) by p^{s-1} and integrating with respect to p between the limits 0 to ∞, we get

$$\int_0^{\infty} C_n^{\alpha}(x,y;ap)\,p^{s-1}dp$$

$$= \frac{a^{\alpha}}{\Gamma(\alpha)\,n!}\int_0^{\infty} t^{\alpha-1}\exp(-at)H_n(2axt,-ayt)$$

$$\times \int_0^{\infty}\exp(-pt^{-1})\,p^{s-1}dp\,dt$$

Using the following relation [7]:

$$\int_0^{\infty}\exp(-pt^{-1})\,p^{s-1}dp = \Gamma(s)\,t^s \;, \quad (2.11)$$

in the R.H.S. of the above equation, we get

$$\int_0^{\infty} C_n^{\alpha}(x,y;ap)\,p^{s-1}dp$$

$$= \frac{a^{\alpha}\Gamma(s)}{\Gamma(\alpha)n!}\int_0^{\infty} t^{\alpha+s-1}\exp(-at)H_n(2axt,-ayt)dt$$

which on using (1.9), we obtain the desired result.

Remark 2.4. Taking $a=1$, $y=1$ and $y=a=1$ respectively in (2.10), we get the following results:

Corollary 2.2. The following Mellin transform representations of $C_n^{\alpha}(x,y;p)$, $C_n^{\alpha}(x;ap)$ and $C_n^{\alpha}(x;p)$ holds true:

$$\int_0^{\infty} C_n^{\alpha}(x,y;p)\,p^{s-1}dp = \frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)} C_n^{\alpha+s}(x,y)\,, \quad (2.12)$$

$$\int_0^{\infty} C_n^{\alpha}(x;ap)\,p^{s-1}dp = \frac{a^{\alpha}\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)} C_n^{\alpha+s}(ax,a,a) \quad (2.13)$$

and

$$\int_0^{\infty} C_n^{\alpha}(x;p)\,p^{s-1}dp = \frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)} C_n^{\alpha+s}(x)\,. \quad (2.14)$$

Theorem 2.3. The following recurrence relation for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;p)$ holds true:

$$(n+1)C_{n+1}^{\alpha}(x,y;p)$$

$$= 2\alpha x\, C_n^{\alpha+1}(x,y;p) - 2\alpha y\, C_{n-1}^{\alpha+1}(x,y;p) \quad (2.15)$$

Proof. Consider the following recurrence relation (see, [2]):

$$H_{n+1}(x,y)=xH_n(x,y)+2nyH_{n-1}(x,y) \qquad (2.16)$$

Replacing x by $2xt$ and y by $-yt$ in (2.16), multiplying both sides by $\frac{1}{\Gamma(\alpha)\,n!}t^{\alpha-1}\exp(-t-pt^{-1})$ and integrating the resultant equation with respect to t between the limits 0 to ∞, we get

$$\frac{1}{\Gamma(\alpha)n!}\int_0^\infty t^{\alpha-1}\exp(-t-pt^{-1})H_{n+1}(2xt,-yt)\,dt$$

$$=\frac{2x}{\Gamma(\alpha)n!}\int_0^\infty t^{\alpha}\exp(-t-pt^{-1})H_n(2xt,-yt)\,dt$$

$$-\frac{2n}{\Gamma(\alpha)n!}\int_0^\infty t^{\alpha}\exp(-t-pt^{-1})H_{n-1}(2xt,-yt)\,dt\,,$$

which by using (2.7) we obtain the desired result.

Remark 2.5. Taking $y=1$ in (2.15), we get the following result:

Corollary 2.3. The following recurrence relation for $C_n^\alpha(x;p)$ holds true:

$$(n+1)C_{n+1}^\alpha(x;p)=2\alpha\,x\,C_n^{\alpha+1}(x;p)-2\alpha\,C_{n-1}^{\alpha+1}(x;p)\,. \qquad (2.17)$$

Theorem 2.4. The following series representation for the new extended Gegenbauer polynomial $C_{2n}^\alpha(x,y;p)$ and $C_n^\alpha(x,y;p)$ holds true:

$$C_{2n}^\alpha(x,y;p)=\frac{2^n(n!)^2}{(2n)!\Gamma(\alpha)}$$

$$\times\sum_{k=0}^{n}\sum_{s=0}^{k}\frac{y^k(-1)^s\Gamma(\alpha+k)(2s)!}{(n-k)!(k-s)!(s!)^2\,2^s}C_{2s}^{\alpha+k}(x,y;p) \qquad (2.18)$$

and

$$C_n^\alpha(x,y;p)=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha-k)(-p)^k}{\Gamma(\alpha)k!}C_n^{\alpha-k}(x,y)\,. \qquad (2.19)$$

Proof of (2.18). Consider the following result [2]:

$$H_{2n}(x,y)=2^n(n!)^2\sum_{k=0}^{n}\frac{[H_k(x,y)]^2}{(k!)^2(n-k)!2^k} \qquad (2.20)$$

Replacing x by $2xt$ and y by $-yt$, in (2.20), multiplying both sides by $t^{\alpha-1}\exp(-t-pt^{-1})$ and integrating with respect to t between the limits 0 to ∞, we get

$$\int_0^\infty t^{\alpha-1}\exp(-t-pt^{-1})H_{2n}(2xt,-yt)\,dt$$

$$=\int_0^\infty t^{\alpha-1}\exp(-t-pt^{-1})\,2^n(n!)^2$$

$$\times\sum_{k=0}^{n}\frac{1}{(k!)^2(n-k)!\,2^k}[H_k(2xt,-yt)]^2\,dt$$

.

Next, using the following result [2]

$$[H_k(x,y)]^2=(-2y)^k(k!)^2\sum_{s=0}^{k}\frac{(-1)^s H_{2s}(x,y)}{(k-s)!(s!)^2\,2^s} \qquad (2.21)$$

and interchanging the order of summation and integration, we obtain

$$\int_0^\infty t^{\alpha-1}\exp(-t-pt^{-1})H_{2n}(2xt,-yt)\,dt$$

$$=2^n(n!)^2\sum_{k=0}^{n}\sum_{s=0}^{k}\frac{y^k(-1)^s}{(n-k)!\,(k-s)!(s!)^2\,2^s}$$

$$\times\int_0^\infty t^{\alpha+k-1}\exp(-t-pt^{-1})H_{2s}(2xt,-yt)\,dt$$

.

Finally, on using (2.7) we obtain the desired result.

Proof of (2.19). From (2.7) we have

$$C_n^\alpha(x,y;p)=\sum_{k=0}^{\infty}\frac{(-p)^k}{k!}$$

$$\times\frac{1}{\Gamma(\alpha)n!}\int_0^\infty t^{\alpha-k-1}\exp(-t)H_n(2xt,-yt)dt\,,$$

which on using (1.9) yields (2.19).

Remark 2.6. Taking $y=1$ in (2.18) and (2.19), we get the following results:

Corollary 2.4. The following series representation for $C_{2n}^\alpha(x;p)$ and $C_n^\alpha(x;p)$ holds true:

$$C_{2n}^\alpha(x;p)=\frac{2^n(n!)^2}{(2n)!\Gamma(\alpha)}$$

$$\times\sum_{k=0}^{n}\sum_{s=0}^{k}\frac{y^k(-1)^s\,\Gamma(\alpha+k)(2s)!}{(n-k)!(k-s)!(s!)^2\,2^s}C_{2s}^{\alpha+k}(x;p) \qquad (2.22)$$

and

$$C_n^\alpha(x;p)=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha-k)(-p)^k}{\Gamma(\alpha)k!}C_n^{\alpha-k}(x)\,. \qquad (2.23)$$

3. Generating functions

In this section, we deduce some generating functions for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;p)$ in the form of the following theorems :

Theorem 3.1. The following generating function for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;p)$ holds true:

$$\sum_{n=0}^{\infty} C_n^{\alpha}(x,y;p)u^n = \frac{(1-2xu+yu^2)^{-\alpha}}{\Gamma(\alpha)}\Gamma_{(1-2xu+yu^2)p}(\alpha) \tag{3.1}$$

where $\mathrm{Re}(1-2xu+u^2)>0$.

Proof. Using (2.7) in the L.H.S. of equation (3.1) and interchanging the order of summation and integration, we get

$$\sum_{n=o}^{\infty} C_n^{\alpha}(x,y;p)u^n = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}\exp(-t-pt^{-1})\sum_{n=o}^{\infty} H_n(2xt,-yt)\frac{u^n}{n!}dt$$

Now, using (1.11), we obtain

$$\sum_{n=o}^{\infty} C_n^{\alpha}(x,y;p)u^n = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}\exp[-t(1-2xu+yu^2)-pt^{-1}]dt \ ,$$

which, by using (1.1), immediately yields the desired result.

Remark 3.1. Taking $y=1$ in (3.1), we get the following result:

Corollary 3.1. The following generating function for $C_n^{\alpha}(x;p)$ holds true:

$$\sum_{n=0}^{\infty} C_n^{\alpha}(x;p)u^n = \frac{(1-2xu+u^2)^{-\alpha}}{\Gamma(\alpha)}\Gamma_{(1-2xu+u^2)p}(\alpha). \tag{3.2}$$

Theorem 3.2. The following generating function for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;p)$ holds true:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+k)!C_{n+k}^{\alpha}(x,y;p)\frac{u^n}{n!}\frac{v^k}{k!} = \frac{\Delta^{-\alpha}}{\Gamma(\alpha)}\Gamma_{\Delta p}(\alpha), \tag{3.3}$$

where $\Delta = 1-2xu+yu^2-2xv+2yuv+yv^2$.

Proof. Using (2.7) in the L.H.S. of equation (3.3) and interchanging the order of summation and integration, we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+k)!C_{n+k}^{\alpha}(x,y;p)\frac{u^n}{n!}\frac{v^k}{k!}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}\exp(-t-pt^{-1})\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} H_{n+k}(2xt,-yt)\frac{u^n}{n!}\frac{v^k}{k!}dt \ .$$

Using the following generating function [4]

$$\sum_{n=0}^{\infty} H_{n+k}(x,y)\frac{u^n}{n!} = \exp(xu+yu^2)H_k(x+2yu,y), \tag{3.4}$$

we obtain

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+k)!C_{n+k}^{\alpha}(x,y;p)\frac{u^n}{n!}\frac{v^k}{k!} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}\exp(-t+2xtu-ytu^2-pt^{-1}) \times\sum_{k=0}^{\infty} H_k(2xt-2ytu,-yt)\frac{v^k}{k!}dt \ .$$

Now, on using (1.11), we obtain

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+k)!C_{n+k}^{\alpha}(x,y;p)\frac{u^n}{n!}\frac{v^k}{k!} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}\exp[-t(1-2xu+yu^2-2xv+2yuv+yv^2)-pt^{-1})\,dt \ ,$$

which, by using (1.1), immediately yields the desired result.

Remark 3.2. Taking $y=1$ in (3.3), we get the following result:

Corollary 3.2. The following generating function for $C_n^{\alpha}(x;p)$ holds true:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(n+k)!C_{n+k}^{\alpha}(x;p)\frac{u^n}{n!}\frac{u^n}{n!} = \frac{(1-2xu+u^2-2xv+2uv+v^2)^{-\alpha}}{\Gamma(\alpha)} \times\Gamma_{(1-2xu+u^2-2xv+2uv+v^2)p}(\alpha) \tag{3.5}$$

Theorem 3.3. The following generating function for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;ap)$ holds true:

$$\sum_{n=0}^{\infty}(1+k)_n C_{n+k}^{\alpha}(x/a,y/a;ap)\frac{u^n}{n!}$$

$$=\frac{a^{\alpha}}{(a-2xu+yu^2)^{\alpha}}$$

$$\times C_k^{\alpha}\left(\frac{x-yu}{a-2xu+yu^2},\frac{y}{a-2xu+yu^2};(a-2xu+yu^2)p\right) \quad (3.6)$$

Proof. In (3.4) replacing x by $2xt$ and y by $-yt$, multiplying both sides by $\frac{1}{\Gamma(\alpha)k!}t^{\alpha-1}\exp(-at-pt^{-1})$ and integrating the resultant equation with respect to t between the limits 0 to ∞, we get

$$\sum_{n=0}^{\infty}\frac{(n+k)!}{a^{\alpha}k!}\frac{a^{\alpha}}{\Gamma(\alpha)(n+k)!}\int_0^{\infty}t^{\alpha-1}\exp(-at-pt^{-1})\,H_{n+k}(2xt,-yt)\frac{u^n}{n!}dt$$

$$=\frac{(a-2xu+yu^2)^{\alpha}}{(a-2xu+yu^2)^{\alpha}\Gamma(\alpha)k!}$$

$$\times\int_0^{\infty}t^{\alpha-1}\exp\left[-(a-2xu+yu^2)t-pt^{-1}\right]H_k(2(x-yu)t,-yt)dt,$$

which by using (2.6), we obtain the desired result.

Remark 3.3. Taking $a=1$ and $y=a=1$ respectively in (3.6), we get the following results:

Corollary 3.3. The following generating functions for $C_n^{\alpha}(x,y;p)$ and $C_n^{\alpha}(x;p)$ hold true:

$$\sum_{n=0}^{\infty}(1+k)_n C_{n+k}^{\alpha}(x,y;p)\frac{u^n}{n!}$$

$$=(1-2xu+yu^2)^{-\alpha}$$

$$C_k^{\alpha}\left(\frac{x-yu}{1-2xu+yu^2},\frac{y}{1-2xu+yu^2};(1-2xu+yu^2)p\right) \quad (3.7)$$

and

$$\sum_{n=0}^{\infty}(1+k)_n C_{n+k}^{\alpha}(x;p)\frac{u^n}{n!}$$

$$=(1-2xu+u^2)^{-\alpha}$$

$$C_k^{\alpha}\left(\frac{x-u}{1-2xu+u^2},\frac{1}{1-2xu+u^2};(1-2xu+u^2)p\right). \quad (3.8)$$

Theorem 3.4. The following generating function for the new extended Gegenbauer polynomials $C_n^{\alpha}(x,y;p)$ holds true:

$$\sum_{n=0}^{\infty}\frac{((d))_n C_n^{\alpha}(x,y;p)t^n}{((g))_n}=\sum_{n=0}^{\infty}\frac{((d))_n(\alpha)_n(2xt)^n}{((g))_n n!}$$

$$\times\,{}_{2D+1}F_{2G}\left[\begin{matrix}(\alpha+n;p),(\frac{1}{2}(d+n)),(\frac{1}{2}(d+n+1)); \\ (\frac{1}{2}(g+n)),(\frac{1}{2}(g+n+1)) \quad ;\end{matrix}-yt^2\right] \quad (3.9)$$

where $((a))_n=(a_1)_n\cdots(a_A)_n=\prod_{j=1}^{A}(a_j)_n$.

Proof. Consider the L.H.S. of equation (3.9) and using (1.4), we get

$$\sum_{n=0}^{\infty}\frac{(d_1)_n\cdots(d_D)_n C_n^{\alpha}(x,y;p)t^n}{(g_1)_n\cdots(g_G)_n}=$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(d_1)_n\cdots(d_D)_n(2x)^{n-2k}(-y)^k(\alpha;p)_{n-k}t^n}{(g_1)_n\cdots(g_G)_n(n-2k)!k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(d_1)_{n+2k}\cdots(d_D)_{n+2k}(2x)^n(-y)^k(\alpha;p)_{n+k}t^{n+2k}}{(g_1)_{n+2k}\cdots(g_G)_{n+2k}n!\,k!}$$

Now using the following results [10] and [13]:

$$(\alpha;p)_{n+k}=(\alpha)_n(\alpha+n;p)_k\ , \mathrm{n,k}\in\mathrm{N}_0\ ,$$

$$(a)_{2n}=2^{2n}\left(\tfrac{1}{2}a\right)_n\left(\tfrac{1}{2}a+\tfrac{1}{2}\right)_n\ ,$$

we get

$$\sum_{n=0}^{\infty}\frac{(d_1)_n\cdots(d_D)_n C_n^{\alpha}(x,y;p)t^n}{(g_1)_n\cdots(g_G)_n}$$

$$=\sum_{n=0}^{\infty}\frac{(d_1)_n\cdots(d_D)_n(\alpha)_n(2xt)^n}{(g_1)_n\cdots(g_G)_n n!}$$

$$\times\sum_{k=0}^{\infty}\frac{(\frac{1}{2}(d_1+n))_k(\frac{1}{2}(d_1+n+1))_k\cdots}{(\frac{1}{2}(g_1+n))_k(\frac{1}{2}(g_1+n+1))_k\cdots}$$

$$\times\frac{(\frac{1}{2}(d_D+n))_k(\frac{1}{2}(d_D+n+1))_k(\alpha+n;p)_k(-yt^2)^k}{(\frac{1}{2}(g_G+n))_k(\frac{1}{2}(g_G+n+1))_k\ k!},$$

which on using the definition (1.6), we obtain the desert result.

Remark 3.4. Taking $y=1$ in (3.9), we get the following result:

Corollary 3.4. The following generating function for $C_n^\alpha(x;p)$ holds true:

$$\sum_{n=0}^{\infty}\frac{((d))_n C_n^\alpha(x;p)t^n}{((g))_n}=\sum_{n=0}^{\infty}\frac{((d))_n(\alpha)_n(2xt)^n}{((g))_n n!}$$

$$\times\ {}_{2D+1}F_{2G}\left[\begin{matrix}(\alpha+n;p),(\tfrac{1}{2}(d+n)),(\tfrac{1}{2}(d+n+1)); \\ (\tfrac{1}{2}(g+n)),(\tfrac{1}{2}(g+n+1)) \quad ;\end{matrix} -t^2\right] \quad (3.10)$$

Theorem 3.5. The following generating functions for the new extended Gegenbauer polynomials $C_n^\alpha(x,y;p)$ holds true:

$$\sum_{n=0}^{\infty}\frac{((d))_n C_n^\alpha(x,y;p)t^n}{((g))_n}$$

$$=\sum_{n=0}^{\infty}\frac{((d))_n(\alpha;p)_n(2xt)^n}{((g))_n n!}\ {}_{D+1}F_G\left[\begin{matrix}-n,(d+n); \\ (g+n) \quad ;\end{matrix}\ \frac{yt}{2x}\right] \quad (3.11)$$

Proof. Consider the L.H.S. of equation (3.11) and using (1.4), we get

$$\sum_{n=0}^{\infty}\frac{(d_1)_n\cdots(d_D)_n C_n^\alpha(x,y;p)t^n}{(g_1)_n\cdots(g_G)_n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(d_1)_n\cdots(d_D)_n(2x)^{n-2k}(-y)^k(\alpha;p)_{n-k}\,t^n}{(g_1)_n\cdots(g_G)_n(n-2k)!k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(d_1)_{n+k}\cdots(d_D)_{n+k}(2xt)^n(-yt^2)^k(\alpha;p)_n}{(g_1)_{n+k}\cdots(g_G)_{n+k}(n-k)!\,k!}$$

$$=\sum_{n=0}^{\infty}\frac{(d_1)_n\cdots(d_D)_n(2xt)^n(\alpha;p)_n}{(g_1)_n\cdots(g_G)_n\ n!}$$

$$\times\sum_{k=0}^{n}\frac{(-n)_k\,(d_1+n)_k\cdots(d_D+n)_k}{(g_1+n)_k\cdots(g_G+n)_k\,k!}\left(\frac{yt}{2x}\right)^k$$

Finally using the definition (1.7), we obtain the desert result.

Remark 3.5. Taking $y=1$ in (3.11), we get the following result:

Corollary 3.5. The following generating function for $C_n^\alpha(x;p)$ holds true:

$$\sum_{n=0}^{\infty}\frac{((d))_n C_n^\alpha(x;p)t^n}{((g))_n}$$

$$=\sum_{n=0}^{\infty}\frac{((d))_n(\alpha;p)_n(2xt)^n}{((g))_n n!}\ {}_{D+1}F_G\left[\begin{matrix}-n,(d+n); \\ (g+n) \quad ;\end{matrix}\ \frac{t}{2x}\right] \quad (3.12)$$

Some special cases of the results (3.9) and (3.11) are as follows:

(i) Setting $D=0, G=1$, $g_1=1$ in (3.9), we get

$$\sum_{n=0}^{\infty}C_n^\alpha(x,y;p)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n(2xt)^n}{(n!)^2}\,{}_1F_2\left[\begin{matrix}(\alpha+n;p) \quad ; \\ \tfrac{1}{2}(n+1),\tfrac{1}{2}(n+2);\end{matrix} -yt^2\right]. \quad (3.13)$$

(ii) Setting $D=G=1, d_1=\lambda$, $g_1=\mu$ in (3.9), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n C_n^\alpha(x,y;p)t^n}{(\mu)_n}$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\alpha)_n(2xt)^n}{(\mu)_n n!}$$

$$\times\ {}_3F_2\left[\begin{matrix}(\alpha+n;p),\tfrac{1}{2}(\lambda+n),\tfrac{1}{2}(\lambda+n+1); \\ \tfrac{1}{2}(\mu+n),\tfrac{1}{2}(\mu+n+1) \quad ;\end{matrix} -yt^2\right], \quad (3.14)$$

which for $y=2x-1$, $\mu=\alpha$, $p=0$ reduces to a known result of Pathan and Khan [8]

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n S_n^\alpha(x)t^n}{(\alpha)_n}=\sum_{n=0}^{\infty}\frac{(\lambda)_n(2xt)^n}{n!}$$

$$\times\ {}_3F_2\left[\begin{matrix}\alpha+n,\tfrac{1}{2}(\lambda+n),\tfrac{1}{2}(\lambda+n+1); \\ \tfrac{1}{2}(\mu+n),\tfrac{1}{2}(\mu+n+1) \quad ;\end{matrix} -(2x-1)t^2\right], \quad (3.15)$$

where $S_n^\alpha(x)$ is the set of polynomials considered by Sinha [11].

(iii) Setting $D=G=0$ in (3.11), we get

$$\sum_{n=0}^{\infty} C_n^{\alpha}(x,y;p)t^n = {}_1F_0\left[\begin{matrix}(\alpha;p); \\ -\;\; ;\end{matrix} 2xt - yt^2\right], \qquad (3.16)$$

which is another form of the generating function (3.1).

(iv) Setting $D = G = 1, d_1 = \lambda$, $g_1 = \mu$ in (3.11) and using the definition of the familiar Lgrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ [6]

$$g_n^{(\alpha,\beta)}(x,y) = {}_2F_1\left[\begin{matrix}-n,\beta & ; \frac{y}{x} \\ 1-\alpha-n; & \end{matrix}\right], \qquad (3.17)$$

we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n C_n^{\alpha}(x,y;p)t^n}{(\mu)_n} = \sum_{n=0}^{\infty} \frac{(\lambda)_n(\alpha;p)_n(2xt)^n}{(\mu)_n(1-\mu-2n)_n} g_n^{(1-\mu-2n,\lambda+n)}(yt,2x), \qquad (3.18)$$

which is another form of the generating function (3.14).

(v) Setting $D = 0, G = 1$, $g_1 = \mu$ in (3.11) and using the definition of the Shively's pseudo Laguerre polynomials $R_n(a,x)$ [9]

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1\left[\begin{matrix}-n\;; \\ a+n;\end{matrix} x\right], \qquad (3.19)$$

we get

$$\sum_{n=0}^{\infty} \frac{C_n^{\alpha}(x,y;p)t^n}{(\mu)_n} = \sum_{n=0}^{\infty} \frac{(\alpha;p)_n(2xt)^n}{(\mu)_{2n}} R_n\left(\mu, \frac{yt}{2x}\right). \qquad (3.20)$$

References

[1] P. Appell, J. Kamp´e de F´eriet, "Fonctions hyper-g´eom´etriques et hypersph´eriques: Polynˆomes d' Her-mite", Gauthier-Villars, Paris, (1926).

[2] C. Cesarano, "Integral representations and new generating functions of Chebyshev polynomials", Hacettepe Journal of Mathematics and Statistics. vol. 44, no. 3, 535 – 546, (2015).

[3] M.A. Chaudhry and S.M. Zubair, "Generalized incomplete gamma functions with applications", J. Comput. Appl. Math., vol. 55, 99-124, (1994).

[4] G. Dattoli, S. Lorenzutta and C. Cesarano, "From Hermite to Humbert polynomials", Rend. Istit. Mat. Univ. Trieste. vol. XXXV, 37-48, (2003).

[5] G. Dattoli, H.M. Srivastava and K. Zhukovsky, "Operational methods and differential equations with applications to initial- value "Applied Mathematics and Computation, vol.180, 979-1001, (2007).

[6] A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, "Higher Transcendental Functions", McGraw-Hill. vols. 1-3, New York, (1953).

[7] I. S. Gradshteyn and I.M. Ryzhik, "Table of Integrals, Series and Products" (edited by Alan Jeffrey, University of New Castle upon Tyne, England), Academic Press, (1994).

[8] M. A. Pathan and M. A. Khan, "On polynomials asso-ciated with Humberts polynomials", Publ. Inst. Math., (Beograd), vol. 62, no. 76, 53-62, (1997).

[9] E. D. Rainville, "Special functions", Macmillan, New York, (1960).

[10] M. Safdar, G. Rahman, Z. Ullah, A. Ghaffar and K.S. Nisar, "A new extension of the Pochhammer symbol and its application to hypergeometric functions", Int. J. Appl. Comput. Math., vol. 5, no. 151, 1-13, (2019).

[11] S. K. Sinha, "On a polynomial associated with Gegenbauer polynomials", Proc. Nat. Acad. Sci., India, vol. 59 (A), no. III, 439-455, (1989).

[12] H. M. Srivastava, A. Çetinkaya and İ Onur Kıyamaz, "A certain generalized Pochhammer symbol and its applications to hypergeometric functions", Appl. Math. Comput. vol. 226, 484-491, (2014).

[13] H. M. Srivastava and H. L. Manocha, "A treatise on generating functions". Halsted Press, New York, (1984).

مقالة بحثية

تمديدات جديدة لمتعددات حدود جيجنبور

أحمد علي عتش[1،*] و أحمد علي الجونة[2]

[1] قسم الرياضيات، كلية التربية شبوة، جامعة عدن، شبوة، اليمن
[2] قسم الرياضيات، كلية العلوم، جامعة عدن، اليمن

*الباحث الممثّل: أحمد علي عتش؛ بريد الكتروني: ah-a-atash@hotmail.com

استلم في: 07 أبريل 2020 / قبل في: 10 يونيو 2020 / نشر في: 30 يونيو 2020

المُلخّص

الهدف الرئيسي لبحثنا هذا هو ادخال تمديدات جديدة لمتعددات حدود جيجنبور ذات المتغير الواحد والمتغيرين وذلك باستخدام تمديد دالة جاما المعطى بواسطة تشودري والزبير **[3]**. تم استنتاج بعض من خواص متعددات الحدود مثل الدوال المولدة والصيغ التكاملية وتحويل ملين.

الكلمات الرّئيسية: دالة جاما الممددة، متعددات جيجنبور الممددة، دوال مولدة، صيغ تكاملية، تحويل ملين.