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RESEARCH ARTICLE

A NEW EXTENSION OF THE HURWITZ- LERCH ZETA FUNCTION AND PROPERTIES USING THE EXTENDED BETA FUNCTION $B_{p,q}^{(\rho,\sigma,\tau)}(x,y)$

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Abstract

The purpose of present paper is to introduce a new extension of Hurwitz-Lerch Zeta function by using the extended Beta function. Some recurrence relations, generating relations and integral representations are derived for that new extension.

Keywords: Extended Beta function, extended Hurwitz-Lerch Zeta function, recurrence relation, generating relation, integral representation.

1. Introduction

Recently, several extensions of Beta function have been investigated (see [1], [5], [6], [8], [12]). In 1997, Chaudhry et al, [6] presented the following extension of Beta function:

$$B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{\left(-\frac{p}{t(1-t)}\right)}dt, \qquad (1.1)$$

$(\mathrm{Re}(p)>0, Re(x)>0, Re(y)>0)$.

The special case p = 0 of (1.1) reduce immediately to classical Beta function [14].

Chaudhry et al. [7] used the extended Beta function given in (1.1) to define the extended Gauss hypergeometric function as follows:

$$F_p(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}. \qquad (1.2)$$

Al-Gonah and Mohommed in [1], introduced the extended Beta function by

$$B_p^{(\rho,\sigma,\tau)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_{\rho,\sigma}^{\tau}\left(\frac{-p}{t(1-t)}\right)dt, \qquad (1.3)$$

$(\mathrm{Re}(p)\geq 0, \mathrm{Re}(\rho)>0, Re(\sigma)>0, Re(\tau)>0, Re(x)>0, Re(y)>0)$,

Also Al-Gonahet al. in [2], introduced extension of the Hurwitz-Lerch Zeta function by using the extension of Beta function defined by (1.3), given by

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{(n+a)^s}, \qquad (1.4)$$

$(\mathrm{Re}(p)\geq 0, \mathrm{Re}(\rho)>0, Re(\sigma)>0, Re(\tau)>0; \lambda,\mu\in\mathbb{C}; \nu, a\in\mathbb{C}\backslash\mathbb{Z}_0^-;$

$s\in\mathbb{C}$ when $|z|<1; Re(s+\nu-\lambda-\mu)>1$ *when* $|z|=1)$.

The following extensions of Beta function are introduced by Atash et al, [3] and Barahmah [5] respectively:

$$B_{p,q}^{(\rho,\sigma)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}E_{\rho,\sigma}\left(-\frac{p}{t}\right)E_{\rho,\sigma}\left(-\frac{q}{(1-t)}\right)dt, \qquad (1.5)$$

$(\mathrm{Re}(p)\geq 0, \mathrm{Re}(q)\geq 0, \mathrm{Re}(\rho)>0, Re(\sigma)>0, Re(x)>0, Re(y)>0)$

And

$$B_{p,q}^{(\rho,\sigma,\tau)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\rho,\sigma}^{\tau}\left(-\frac{p}{t}\right) E_{\rho,\sigma}^{\tau}\left(-\frac{q}{(1-t)}\right) dt, \quad (1.6)$$

$(\mathrm{Re}(p) \geq 0, \mathrm{Re}(q) \geq 0, \mathrm{Re}(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0, Re(x) > 0, Re(y) > 0),$

where $E_{\rho,\sigma}(z)$ and $E_{\rho,\sigma}^{\tau}(z)$denotes the generalized Mittag-Leffler functions defined by [15] and [10] as follows:

$$E_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)}, \quad (1.7)$$

$(z, \rho, \sigma, \tau \in \mathbb{C}; Re(\rho) > 0, Re(\sigma) > 0)$

And

$$E_{\rho,\sigma}^{\tau}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!}, \quad (1.8)$$

$(z, \rho, \sigma, \tau \in \mathbb{C}; \mathrm{Re}(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0).$

The extended Beta function given in equations (1.5) and (1.6) are used in [4] to define the following extended Gauss hypergeometric function:

$$F_{p,q}^{(\rho,\sigma)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}^{(\rho,\sigma)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad (1.9)$$

$(Re(p) \geq 0, Re(p) \geq 0, |z| < 1; Re(c) > Re(b) > 0, Re(\rho) > 0, Re(\sigma) > 0)$

and

$$F_{p,q}^{(\rho,\sigma,\tau)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}^{(\rho,\sigma,\tau)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad (1.10)$$

$(\mathrm{Re}(p) \geq 0, \mathrm{Re}(p) \geq 0, |z| < 1; Re(c) > Re(b) > 0$

$\mathrm{Re}(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0)$.

Clearly,

$$F_{p,q}^{(\alpha,\beta,1)} = F_{p,q}^{(\alpha,\beta)},\ F_{p,p}^{(1,1,1)} = F_p,\ F_{0,0}^{(1,1,1)} = {}_2F_1.$$

The Hurwitz-Lerch Zeta function$\Phi(z,s,a)$ is defined by (see, e,g,, [13, p. 121])

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}, \quad (1.11)$$

$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ s \in C \text{ when } |z| < 1;\ Re(s) > 1 \text{ when } |z| = 1).$

The following generalizations of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$is defined in [9, p. 313] as:

$$\Phi_{\lambda,\mu;\nu}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{n!\ \ (\nu)_n} \frac{z^n}{(n+a)^s}, \quad (1.12)$$

$(\lambda, \mu, \nu \in \mathbb{C}; a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = 1).$

The integral representation of (1.12) is given by:

$$\Phi_{\lambda,\mu;\nu}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at}\, {}_2F_1(\lambda, \mu; \nu; ze^{-t})\, dt. \quad (1.13)$$

Parmar and Raina [11] used the extended Beta function $B(x,y;p)$ to introduce the following extension of generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B(\mu+n, \nu-\mu;p)}{B(\mu, \nu-\mu)} \frac{z^n}{(n+a)^s}, \quad (1.14)$$

$(p \geq 0; \lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \mathrm{Re}(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = 1),$

with the integral representation [11, p. 120(3,1)]:

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} F_p(\lambda, \mu; \nu; ze^{-t})\, dt, \quad (1.15)$$

$(\mathrm{Re}(p) \geq 0; p = 0, \mathrm{Re}(a) > 0; Re(s) > 0 \text{ when } |z| \leq 1$
$(z \neq 1); \mathrm{Re}(s) > 1 \text{when } z = 1)$

Motivated by various recent interesting extensions of the Hurwitz-Lerch Zeta function, we introduce a new form of extended Hurwitz-Lerch Zeta function by using the extension of Beta function $B_{p,q}^{(\rho,\sigma,\tau)}(x,y)$and we investigate its properties.

2. A new extended Hurwitz-Lerch Zeta function

We consider the following generalization of extended Hurwitz-Lerch Zeta function

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+n, \nu-\mu)}{B(\mu, \nu-\mu)} \frac{z^n}{(n+a)^s}, \quad (2.1)$$

$\left(\begin{array}{c}\mathrm{Re}(p) \geq 0, \mathrm{Re}(q) \geq 0, \mathrm{Re}(\rho) > 0, Re(\sigma) > 0, \\ Re(\tau) > 0; \lambda, \mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; \\ s \in \mathbb{C} \text{ when } |z| < 1; Re(s + \nu - \lambda - \mu) > 1 \\ \text{when } |z| = 1\end{array}\right).$

Remark 2.1.

(i) For $\tau = 1$, equation (2.1) reduces to the following new extended Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p,q)$

$$\Phi^{(\rho,\sigma,1)}_{\lambda,\mu;\nu}(z,s,a;p,q) = \Phi^{(\rho,\sigma)}_{\lambda,\mu;\nu}(z,s,a;p,q)$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B^{(\rho,\sigma)}_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^n}{(n+a)^s}, \qquad (2.2)$$

$$\left(\begin{matrix} \mathrm{Re}(p) \geq 0, \mathrm{Re}(q) \geq 0, \mathrm{Re}(\rho) > 0, Re(\sigma) > 0; \\ \lambda,\mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; \\ s \in \mathbb{C} \text{ when } |z| < 1; Re(s+\nu-\lambda-\mu) > 1 \text{ when } |z| = 1 \end{matrix}\right).$$

(ii) For $\tau = \sigma = \rho = 1$ and $p = q$, equation (2.1) reduces to theextended Hurwitz-Lerch Zeta function given in (1.14)

$$\Phi^{(1,1,1)}_{\lambda,\mu;\nu}(z,s,a;p,p) = \Phi_{\lambda,\mu;\nu}(z,s,a;p)$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B_p(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^n}{(n+a)^s}. \qquad (2.3)$$

(iii) For $\tau = \sigma = \rho = 1$ and $p = q = 0$, equation (2.1) reduces to the extended Hurwitz-Lerch Zeta function given in (1.12)

$$\Phi^{(1,1,1)}_{\lambda,\mu;\nu}(z,s,a) = \Phi_{\lambda,\mu;\nu}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{n! \ (\nu)_n} \frac{z^n}{(n+a)^s}. \qquad (2.4)$$

Remark 2.2. The extended Hurwitz-Lerch function $\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q)$ defined by equation (2.1) is seen to satisfy the following limit case:

$$\Phi^{*(\rho,\sigma,\tau)}_{\mu;\nu}(z,s,a;p,q) = \lim_{|\lambda|\to\infty} \left\{\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}\left(\frac{z}{\lambda},s,a;p,q\right)\right\},$$

$$= \sum_{n=0}^{\infty} \frac{B^{(\rho,\sigma,\tau)}_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^n}{n!\,(n+a)^s}, \qquad (2.5)$$

$$\left(\begin{matrix} \mathrm{Re}(p) \geq 0, \mathrm{Re}(p) \geq 0, \mathrm{Re}(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0; \\ \mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; \\ s \in \mathbb{C} \text{ when } |z| < 1; Re(s+\nu-\mu) > 1 \\ when \ |z| = 1 \end{matrix}\right).$$

Some properties of the extended Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q)$ are established in the form of the following theorems

Theorem 2.1. For the extended Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q)$, we have the following recurrence relation:

$$\nu\, \Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q) = \mu\, \Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu+1;\nu+1}(z,s,a;p,q) + (\nu-\mu)\, \Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu+1}(z,s,a;p,q), \qquad (2.6)$$

$$(\mathrm{Re}(p) \geq 0, \mathrm{Re}(q) \geq 0, \mathrm{Re}(\nu) > Re(\mu) > 0).$$

Proof. To prove (2.6) using the following known relation [5]:

$$B^{(\rho,\sigma,\tau)}_{p,q}(x,y) = B^{(\rho,\sigma,\tau)}_{p,q}(x+1,y) + B^{(\rho,\sigma,\tau)}_{p,q}(x,y+1), \qquad (2.7)$$

in definition (2.1), we get:

$$\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \times \left[\frac{B^{(\rho,\sigma,\tau)}_{p,q}(\mu+n+1,\nu-\mu) + B^{(\rho,\sigma,\tau)}_{p,q}(\mu+n,\nu-\mu+1)}{B(\mu,\nu-\mu)}\right] \times \frac{z^n}{(n+a)^s}$$

$$= \frac{B(\mu+1,\nu-\mu)}{B(\mu,\nu-\mu)} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B^{(\rho,\sigma,\tau)}_{p,q}(\mu+n+1,\nu-\mu)}{B(\mu+1,\nu-\mu)} \frac{z^n}{(n+a)^s}$$

$$+ \frac{B(\mu,\nu-\mu+1)}{B(\mu,\nu-\mu)} \times \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B^{(\rho,\sigma,\tau)}_{p,q}(\mu+n,\nu-\mu+1)}{B(\mu,\nu-\mu+1)} \frac{z^n}{(n+a)^s}, \qquad (2.8)$$

which on using the relation [14]:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (x,y \in \mathbb{C}\backslash\mathbb{Z}_0^-) \qquad (2.9)$$

and then using definition (2.1) gives the desired result.

Remark 2.3. Using the following relations [5,p. 44]:

$$B^{(\rho,\sigma,\tau)}_{p,q}(x,y) = \sum_{n=0}^{\infty} B^{(\rho,\sigma,\tau)}_{p,q}(x+n,y+1), \qquad (2.10)$$

$$B^{(\rho,\sigma,\tau)}_{p,q}(x,1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B^{(\rho,\sigma,\tau)}_{P,q}(x+n,1), \qquad (2.11)$$

$$B^{(\rho,\sigma,\tau)}_{p,q}(x,y) = \sum_{n=0}^{k} \binom{k}{n} B^{(\rho,\sigma,\tau)}_{p,q}(x+n,y+k-n), \quad (k \in \mathbb{N}), \qquad (2.12)$$

Theorem 2.2. For the extended Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q)$, we have the following summation relations:

$$\Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu;\nu}(z,s,a;p,q) = (\nu-\mu) \sum_{k=0}^{\infty} \frac{(\mu)_k}{(\nu)_{k+1}} \Phi^{(\rho,\sigma,\tau)}_{\lambda,\mu+k;\nu+k+1}(z,s,a;p,q), \qquad (2.13)$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q) = \sum_{k=0}^{\infty} \frac{(\mu-\nu+1)_k B(\mu+k,1)}{k!\,B(\mu,\nu-\mu)} \Phi_{\lambda,\mu+k;\mu+k+1}^{(\rho,\sigma,\tau)}(z,s,a;p,q), \quad (2.14)$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q) = \sum_{k=0}^{r} \binom{r}{k} \frac{B(\mu+k,\nu-\mu-k+r)}{B(\mu,\nu-\mu)} \Phi_{\lambda,\mu+k;\nu+r}^{(\rho,\sigma,\tau)}(z,s,a;p,q) \quad (2.15)$$

and proceeding on the same lines of proof of Theorem 2.1. we get some summation relations in the form of the following theorem:

Theorem 2.3. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, we have the following derivative formula:

$$\frac{d^k}{dz^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)\right\} = \frac{(\lambda)_k(\mu)_k}{(\nu)_k} \Phi_{\lambda+k,\mu+k;\nu+k}^{(\rho,\sigma,\tau)}(z,s,a+k;p,q), \quad (k \in \mathbb{N}_0). \quad (2.16)$$

Proof. From definition (2.1) and using the following relation [15]:

$$\frac{d^k}{dz^k} z^n = \frac{n!}{(n-k)!}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)\right\} = \frac{(\lambda)_k(\mu)_k}{(\nu)_k} z^{n-k}, (k \in \mathbb{N}_0), \quad (2.17)$$

we fined

$$\frac{d^k}{dz^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)\right\}$$

$$= \sum_{n=k}^{\infty} \frac{(\lambda)_n}{(n-k)!} \frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^{n-k}}{(n+a)^s}, \quad (2.18)$$

replacing n by $n+k$, we obtain

$$\frac{d^k}{dz^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)\right\}$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_{n+k}}{(n)!} \frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+n+k,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^n}{(n+k+a)^s}, \quad (2.19)$$

which on using the following relations [14]:

$$B(b,c-b) = \frac{(c)_k}{(b)_k} B(b+k,c-b), \quad (2.20)$$

$$(a)_{m+n} = (a)_m(a+m)_n, \quad (2.21)$$

and then in view of definition (2.1), we get the desired result.

3. Generating relations

In this section, some generating functions for the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$ is established in the form of the following theorem:

Theorem 3.1. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, the following generating function holds true:

$$\sum_{n=0}^{\infty} (\lambda)_n \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)\frac{t^n}{n!} = (1-t)^{-\lambda}\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}\left(\frac{z}{1-t},s,a;p,q\right), \quad (3.1)$$

$$(Re(p) \geq 0, Re(q) \geq 0, \lambda \in \mathbb{C}, |t| < 1).$$

Proof. Denoting the L.H.S. of equation (3.1) by Δ then applying definition (2.1), we get:

$$\Delta = \sum_{n=0}^{\infty} (\lambda)_n \times \left\{\sum_{k=0}^{\infty} (\lambda+n)_k \frac{B_{p,q}^{(\rho,\sigma,\tau)} B(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^k}{k!(k+a)^s}\right\} \frac{t^n}{n!}, \quad (3.2)$$

which on using relation (2.21), we obtain

$$\Delta = \sum_{k=0}^{\infty} (\lambda)_k \frac{B_{p,q}^{(\rho,\sigma,\tau)} B(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{z^k}{k!\,(k+a)^s} \times \left\{\sum_{n=0}^{\infty} \frac{(\lambda+k)_n}{n!} t^n\right\} \quad (3.3)$$

using the following binomial series expansion [14]:

$$(1-t)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} t^n, \qquad (|t| > 1), \quad (3.4)$$

for evaluating the inner sum in equation (3.3), we get the desired result,

Theorem 3.2. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, the following generating function holds true:

$$\sum_{n=0}^{\infty} (s)_n \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s+n,a;p,q)\frac{t^n}{n!} = \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a-t;p,q), \quad (3.5)$$

$$(Re(p) \geq 0, Re(q) \geq 0, \lambda \in \mathbb{C}, |t| < 1|a|; s \neq 1).$$

Proof. Using definition (2.1) in the R.H.S. of equation (3.5), we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a-t;p,q)$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a-t)^s},$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^s}\times\left(1-\frac{t}{(k+a)}\right)^{-s}, \quad (3.6)$$

using relation (3.4), we obtain

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a-t;p,q)$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^s}\times\left\{\sum_{n=0}^{\infty}(s)_n\frac{t^n}{n!\,(k+a)^n}\right\},$$
$$=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\left\{\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^{s+n}}\right\}t^n, \quad (3.7)$$

which on using definition (2.1) and after some simplification yields the desired result,

Theorem 3.3. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, the following generating function holds true:

$$\sum_{n=0}^{\infty}\Phi_{-n,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)\frac{t^n}{n!}=e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p,q), \quad (3.8)$$

$(Re(p)\geq 0, Re(q)\geq 0, Re(\nu)>Re(\mu)>0;$ $Re(\rho), Re(\sigma), Re(\tau)>0),$

Proof. Using relation (2.5) in the R.H.S. of equation (3.8), we get:

$$e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p,q)$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(-1)^k z^k}{k!\,(k+a)^s}\frac{t^{n+k}}{n!}, \quad (3.9)$$

replacing n by $n-k$ in the R.H.S. of equation (3.9), we get :

$$e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p,q)$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)(n-k)!}\frac{(-1)^k t^n z^k}{k!\,(k+a)^s}, \quad (3.10)$$

using the following relation [14]:

$$(n-k)!=\frac{(-1)^k n!}{(-n)_k}, \qquad (0\leq n\leq k), \quad (3.11)$$

in equation (3.10), we obtain

$$e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p,q)$$
$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty}\frac{(-n)_k}{k!}\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^s}\right\}\frac{t^n}{n!}, \quad (3.12)$$

which on using definition (2.1) yields the desired result.

4. Integral representations

In this section, some integral representations for the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$ are established in the form of the following theorem:

Theorem 4.1. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, the following integral representation holds true:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$$
$$=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\,e^{-at}F_{p,q}^{(\rho,\sigma,\tau)}(\lambda,\mu;\nu;ze^{-t})dt, \quad (4.1)$$

$(Re(p)\geq 0, Re(q)\geq 0, Re(\rho)>0, Re(\sigma)>0, Re(\tau)>0; p=0,$

$Re(a)>0; Re(s)>0\ when|z|\leq 1\ (z\neq 1); Re(s)>1\ whenz=1),$

Proof. Using the Eulerian integral [14,p. 218(3)]:

$$\frac{1}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\,e^{-(n+a)t}dt,$$

$$(min\{Re(s),Re(a)\}>0; n\in\mathbb{N}_0), \quad (4.2)$$

in definition (2.1) and interchanging the order of summation and integration which may be valid under the conditions stated in Theorem 4.1. we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\,e^{-at}$$
$$\times\left(\sum_{n=0}^{\infty}(\lambda)_n\frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(ze^{-t})^n}{n!}\right)\mathrm{dt}, \quad (4.3)$$

which on using definition (1.9) gives the desired result,

Theorem 4.2. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, the following integral representation holds true:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)=\frac{1}{B(\mu,\nu-\mu)}\int_0^\infty \frac{u^{\mu-1}}{(1+u)^\nu}$$

$$\times\, E_{\rho,\sigma}^{\tau}\left(-p\left(\frac{1+u}{u}\right)\right)E_{\rho,\sigma}^{\tau}\left(\frac{-q}{1+u}\right)\Phi_\lambda^*\left(\frac{zu}{1+u},s,a\right)du, \quad (4.4)$$

$(Re(p)\geq 0, Re(q)\geq 0, Re(\rho)>0, Re(\sigma)>0,$
$Re(\tau)>0; p=0, Re(\nu)>Re(\mu)>0)$
and

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)=\frac{1}{\Gamma(s)B(\mu,\nu-\mu)}$$
$$\times\int_0^\infty\int_0^\infty \frac{t^{s-1}e^{-at}u^{\mu-1}}{(1+u)^\nu}E_{\rho,\sigma}^{\tau}\left(-p\left(\frac{1+u}{u}\right)\right)$$
$$E_{\rho,\sigma}^{\tau}\left(\frac{-q}{1+u}\right)\left(1-\frac{zue^{-t}}{1+u}\right)^{-\lambda}dtdu, \quad (4.5)$$

$(Re(p)\geq 0, Re(q)\geq 0, Re(\rho)>0, Re(\sigma)>0,$
$Re(\tau)>0; p=0,$
$Re(\nu)>Re(\mu)>0, min\{Re(s), Re(a)\}>0)$

Proof. Putting $x=\mu+n$ and $y=\nu-\mu$ in the following integral representation of the extended Beta function [5,p. 43(1.13)]:

$$B_{p,q}^{(\rho,\sigma,\tau)}(x,y)=\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}$$
$$\times\, E_{\rho,\sigma}^{\tau}\left(-p\left(\frac{1+u}{u}\right)\right)E_{\rho,\sigma}^{\tau}\left(\frac{-q}{1+u}\right)du, \quad (4.6)$$

we obtain

$$B_{p,q}^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)=\int_0^\infty \frac{u^{\mu+n-1}}{(1+u)^{\nu+n}}$$
$$\times\, E_{\rho,\sigma}^{\tau}\left(-p\left(\frac{1+u}{u}\right)\right)E_{\rho,\sigma}^{\tau}\left(\frac{-q}{1+u}\right)du, \quad (4.7)$$

which on using it in definition (2.1) yields

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)=\frac{1}{B(\mu,\nu-\mu)}\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}$$

$$\times\int_0^\infty \frac{u^{\mu+n-1}}{(1+u)^{\nu+n}}E_{\rho,\sigma}^{\tau}\left(-p\left(\frac{1+u}{u}\right)\right)E_{\rho,\sigma}^{\tau}\left(\frac{-q}{1+u}\right)$$

$$\times\frac{z^n}{(n+a)^s}du, \quad (4.8)$$

interchanging the order of summation and integration in equation (4.8), which is verified under the given conditions here, we get:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)=\frac{1}{B(\mu,\nu-\mu)}\int_0^\infty \frac{u^{\mu-1}}{(1+u)^\nu}$$

$$\times\, E_{\rho,\sigma}^{\tau}\left(-p\left(\frac{1+u}{u}\right)\right)E_{\rho,\sigma}^{\tau}\left(\frac{-q}{1+u}\right)$$
$$\times\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\frac{\left(\frac{zu}{1+u}\right)^n}{(n+a)^s}du, \quad (4.9)$$

using definition (1.11) in the R.H.S. of equation (4.9), we get the desired result (4.4).

Also, using definition (1.18) in the R.H.S. of equation (4.4), we get the desired result (4.5) and thus the proof of Theorem 4.2 is completed.

Theorem 4.3. For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$, the following integral representation holds true:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$$
$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\Phi_{\lambda,\mu;\nu}^{*(\rho,\sigma,\tau)}(zt,s,a;p,q)dt, \quad (4.10)$$

$(Re(p)\geq 0, Re(q)\geq 0, Re(\rho)>0, Re(\sigma)>0,$
$Re(\tau)>0; p=0, Re(\lambda)>0,$
$Re(a)>0; Re(s)>0\ when |z|\leq 1\ (z\neq 1);$
$Re(s)>1\ when z=1).$

Proof. Applying the following integral representation of the Pochhammer symbol $(\lambda)_n$:

$$(\lambda)_n=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+n-1}e^{-t}dt, \quad (4.11)$$

in definition (2.1) and inverting the order of summation and integration which may be permissible under the conditions stated Theorem 4.3. we get:
$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\sum_{n=0}^\infty \frac{B_{p,q}^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}$$
$$\times\frac{(zt)^n}{n!\ (n+a)^s}dt, \quad (4.12)$$

which on using equation (2.5) gives the desired result.

Remark 4.1.
(1) If we set $\tau=1$in (2.6), (2.13) -(2.16), (3.1),(3.5),(3.8),(4.1),(4.4),(4.5),(4.10), we obtain a new results for extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}(z,s,a;p,q)$ given in (2.2).

(2) If we set $\rho=\sigma=\tau=1$, $p=q$in the above mentioned results,we obtain a known corresponding results due to Parmar and Raina [11] for the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$given in (1.14).

5. Conclusion

In this paper, new extension of Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p,q)$ is and new results can be obtained as special cases of the main results obtained in the previous introduced with the help of the extended Beta function $B_{p,q}^{(\rho,\sigma,\tau)}(x,y)$ given in [5]. Various properties of that extended function are investigated such as recurrence relation, generating relations and integral representations, it is interesting to mention here that many know sections.

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مقالة بحثية

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