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RESEARCH ARTICLE

A NEW EXTENSION OF LOGARITHMIC BETA FUNCTION AND THEIR PROPERTIES

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Abstract

Khan Et Al., presented a new kind of beta logarithmic function, we aim in this research article to introduce new extension of beta logarithmic function, Further, we study its fundamental properties and discuss diverse formulas of that extension such as integral representation, summation formula, transform formula and their statistical properties. Based on this concept, we introduce new hypergeometric and confluent hypergeometric functions and study their properties.

Keywords: Logarithmic mean, Beta function, Hypergeometric function, Confluent hypergeometric function.

1. Introduction and Preliminaries

Here, we introduce and investigate new extension of properties and representations of a extension of beta function known as beta logarithmic function a combined study of beta function (1.18) and logarithmic mean (1.1). The logarithmic mean defined as;

$$L(c,d)=\int_0^1 c^{1-t}\ d^t\ d$$

$$=\begin{cases}\dfrac{c-d}{\log(c)-\log(d)} & c\neq d\\ c & c=d\end{cases} \qquad (1.1)$$

Classic Gaussian hypergeometric functions (see [1])

are define

$$F(x,y;\omega;z)=\sum_{n=0}^{\infty}\frac{(x)_n(y)_n}{(\omega)_n}\frac{z^n}{n!}, \qquad (1.2)$$

and the confluent hypergeometric functions (see [1]) is defined by

$$\Phi(y;\omega;z)=\sum_{n=0}^{\infty}\frac{(y)_n}{(\omega)_n}\frac{z^n}{n!}. \qquad (1.3)$$

where $(\mu)_n$ $(\mu\in\mathbb{C})$ is the Pochhammer symbol defined by

$$(\mu)_n=\frac{\Gamma(\mu+n)}{\Gamma(\mu)}. \qquad (1.4)$$

The Euler beta function $B(x,y)$ (see [1]) is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\ dt \qquad (1.5)$$

$$=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\frac{(x-1)!\ (y-1)!}{(x+y-1)!}, \qquad (1.6)$$

where $(\Re(x)>0\ ,\Re(y)>0)$.

In [4], introduced an extension of beta function defined by

$$B^p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,exp\left(-\frac{p}{t(1-t)}\right)dt, \qquad (1.7)$$

where $\Re(p)\geq 0$, $(\Re(x)>0\ ,\mathcal{R}(y)>0)$.

In [5], used new extended beta function $B^p(\alpha,\delta)$ to introduced an extended hypergeometric and confluent hypergeometric function defined respectively as

$$F^p(x,y,\omega;z)=\sum_{n=0}^{\infty}(\delta_1)_n\frac{B^p(x+n,\omega-y)}{B(y,\omega-y)}\frac{z^n}{n!}, \qquad (1.8)$$

$$(p\geq 0,\ |z|<1,\ \Re(x)>\Re(y)>0)$$

and

$$\Phi^p(y;\omega;z)=\sum_{n=0}^{\infty}\frac{B^p(y+n,\omega-y)}{B(y,\omega-y)}\frac{z^n}{n!}, \qquad (1.9)$$

The integral representation for extended Gauss hypergeometric functions and extended confluent hypergeometric functions are defined as:

$$F^{p}(x,y,\omega;z)=\frac{1}{B(y,\omega-y)}\int_{0}^{1}t^{y-1}\,(1-t)^{\omega-y-1}(1-zt)^{x}exp\left(-\frac{p}{t(1-t)}\right)dt, \quad (1.10)$$

$(p\geq 0\,;\ \ |arg\,(1-z)\,|<1;\ \ \Re(\omega)>\Re(y)>0).$

$$\Phi^{p}(y;\omega;z)=\frac{1}{B(y,\omega-y)}\int_{0}^{1}t^{y-1}\,(1-t)^{\omega-y-1}exp\left(zt-\frac{p}{t(1-t)}\right)dt \quad (1.11)$$

$(p\geq 0\,;\quad \Re(\omega)>\mathcal{R}(y)>0).$

In [9], introduced an extension of beta function using generalized Mittag-Leffler function as follow:

$$B_{\alpha,\beta}^{p,\mu,\nu}(x,y)=\int_{0}^{1}t^{x-1}\,(1-t)^{y-1}E_{\alpha,\beta}\left(-\frac{p}{t^{u}(1-t)^{v}}\right)dt, \quad (1.12)$$

$\Re(p)>0,\Re(x)>0\,,\Re(y)>0,$
$\alpha,\beta\in\mathbb{R}_{0}^{+}\,,\mu,\nu\in\mathbb{R}^{+}.$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined as [16]

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+\beta)}\,, \quad (1.13)$$

where $x\in\mathbb{C}\ ,\alpha,\beta\in\mathbb{R}_{0}^{+}$.

By using (1.12), we further extended the Gauss hypergeom- etric and confluent hypergeometric functions and their integral representation are defined as

$$F_{\alpha,\beta}^{(p,\mu,\nu)}(x,y;\omega;z)=\sum_{n=0}^{\infty}(x)_{n}\,\frac{B_{\alpha,\beta}^{(p,\mu,\nu)}(x+n,\omega-y)}{B(y,\omega-y)}\,\frac{z^{n}}{n!}, \quad (1.14)$$

$(p\geq 0\,,|z|<1\,,\alpha,\beta,\mu,\nu\ \in\mathbb{R}^{+}\,,\ \ \Re(\omega)>\Re(y)>0).$

$$\Phi_{\alpha,\beta}^{(p,\mu,\nu)}(y;\omega;z)=\sum_{n=0}^{\infty}\frac{B_{\alpha,\beta}^{(p,\mu,\nu,\gamma,\sigma)}(y+n,\omega-y)}{B(y,\omega-y)}\frac{z^{n}}{n!}, \quad (1.15)$$

$(p\geq 0\,,\ \ \alpha,\beta,\mu,\nu\ \in\mathbb{R}^{+}\,,\ \ \Re(\omega)>\Re(\delta_{2})>0).$

$$F_{\alpha,\beta}^{(p,\mu,\nu)}(x,y,\omega;\tau)=\frac{1}{B(\delta_{2},\omega-\delta_{2})}\int_{-1}^{1}t^{y-1}(1-t)^{\omega-y-1}\times(1-zt)^{-x}\,E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)dt, \quad (1.16)$$

$(p\in\mathbb{R}_{0}^{+},\alpha,\beta,\gamma,\sigma,\mu,\nu\in\mathbb{R}^{+}\,;\ and\ \ arg|1-z|<\pi,\ \Re(\omega)>\Re(\delta_{2})>0)$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,\nu)}(y;\omega;\tau)=\frac{1}{B(y,\omega-y)}\int_{-1}^{1}t^{y-1}(1-t)^{\omega-y-1}\,e^{zt}\times E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)dt, \quad (1.17)$$

$(p\in\mathbb{R}_{0}^{+},\ \ \alpha,\beta,\gamma,\sigma,\mu,\nu\in\mathbb{R}^{+}\,;\quad Re(\omega)>Re(y)>0)$

In [2], introduced a new extended Beta function in terms of the classical Mittag-Leffler function defined as

$$B_{\alpha,\beta}^{(p,\mu,\nu,\gamma,\sigma)}(x,y)=\int_{0}^{1}t^{x-1}\,(1-t)^{y-1}\times E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}(1-t)^{v}}\right)dt, \quad (1.18)$$

$\Re(p)>0,\ \ \Re(\delta_{1})>0\,,\Re(\delta_{2})>0,\ \ \alpha,\beta,\gamma,\sigma\in\mathbb{R}_{0}^{+}\,,$
$\mu,\nu\in\mathbb{R}^{+}.$

where $E_{\alpha,\beta}^{\gamma,\sigma}(.)$ is the generalized Mittag-Leffler function defined as [15]

$$E_{\alpha,\beta}^{\gamma,\rho}\,(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_{\rho k}}{\Gamma(\alpha k+\beta)}\,\frac{z^{k}}{k!} \quad (1.19)$$

where $\alpha,\beta,\gamma\in\mathbb{C},\ \Re\,(\alpha),\ \Re\,(\beta)\,,\ \Re\,(\gamma)>0,$
$\rho\ \in\ (0,1)\ \cup\ \mathbb{N}.$

Recently, Khan et al. [9] introduced the following definition

$$B\,L_{\alpha,\beta}^{(p,\mu,\nu)}(c,d;x,y)=\int_{0}^{1}c^{1-t}\,d^{t}\,t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}\left(-\frac{p}{t^{u}(1-t)^{v}}\right)dt, \quad (1.20)$$

2. Main results

In this section, we will introduce the new extension of the beta logarithmic function(1.20) and study its properties and representations.

For any fixed $c,d>0$ the function $t\to c^{1-t}\,d^{t}$ is continous in $[0,1]$ and so it is bounded on $[0,1]$. It means that there exist $h\geq 0$ and for any $c,d,x,y>0$, we have

$$0\leq c^{1-t}\,d^{t}\,t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}(1-t)^{v}}\right)$$
$$\leq h\,t^{x-1}(1-t)^{y-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}(1-t)^{v}}\right)$$

$$\forall\, t \in (0,1)$$

Thus, $c^{1-t}\, d^t\, t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)$ is integrable on $(0,1)$.Then we intr- oduce the following definition:

Definition 2.1. For any $\alpha, \beta, \mu, v, \gamma, \sigma, c, d \in \mathbb{R}^+$, we define

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = \int_0^1 c^{1-t}\, d^t\, t^{x-1}(1-t)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt \qquad (2.1)$$

$$(p \geq 0\,, \quad \Re(x) > 0\,, \qquad \Re(y) > 0).$$

where $E_{\alpha,\beta}^{\gamma,\sigma}(.)$ is the generalized Mittag-Leffler function defined as (1.19).

If we take $\gamma = 1$ *and* $\sigma = 1$ in (2.1), we get the beta function introduced by (see [10]) i.e.

$$B\, L_{\alpha,\beta}^{(p,\mu,v,1,1)}(c,d;x,y) = B\, L_{\alpha,\beta}^{(p,\mu,v)}(c,d,x,y) \qquad (2.2)$$

If we take $c = 1$ *and* $d = 1$ in (2.1), we get the new result of beta function

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(1,1;x,y) = B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y) \qquad (2.3)$$

If $c = d = \sigma = \gamma = 1$, in Eq. (2.1), we get the beta function introduced by (see [9, 11]) i.e.

$$B\, L_{\alpha,\beta}^{(p,\mu,v,1,1)}(1,1;x,y) = B\, L_{\alpha,\beta}^{(p,\mu,v)}(x,y) \qquad (2.4)$$

If we take $\alpha = \beta = \mu = v = \sigma = \gamma = c = d = 1$ *and* $p = 0$ in (2.1), we get an Euler beta functions (1.5) (see [1, 6])

$$B\, L_{1,1}^{(0,1,1,1,1)}(1,1;x,y) = B(x,y) \qquad (2.5)$$

If we choose $\alpha = \beta = \mu = v = \sigma = \gamma = x = y = 1$ *and* $p = 0$ in (2.1), then we get a logarithmic mean (1.1)

$$B\, L_{1,1}^{(0,1,1,1,1)}(c,d;1,1) = L(c,d) \qquad (2.6)$$

3. Properties of $BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)$

In this section we obtain some assertions for $B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)$

Theorem 3.1. For $\alpha, \beta, c, d, x, y > 0$, the following assertions holds true:

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(d,c;x,y) \qquad (3.1)$$

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,c;x,y) = c\, B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y) \qquad (3.2)$$

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\xi c, \xi d;x,y) = \xi B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(d,c;x,y) \qquad (3.3)$$

Proof: Let $t = 1-u$ in equation (2.1), we obtain the above assertions.

Theorem 3.2. For $\alpha, \beta, c, d, x, y > 0$, the following assertions holds true:

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+1,y) + B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y+1) = B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) \qquad (3.4)$$

Proof: By using the definition (2.1) to the left side of (3.4), we get the assertion (3.4).

Remark 3.1. If we take $\gamma = 1$ *and* $\sigma = 1$ in (3.4), we get the beta function introduced by Khan et al. (see [10]) i.e.

$$B\, L_{\alpha,\beta}^{(p,\mu,v)}(c,d;x+1,y) + B\, L_{\alpha,\beta}^{(p,\mu,v,)}(c,d;x,y+1) = B\, L_{\alpha,\beta}^{(p,\mu,v,)}(c,d;x,y) \qquad (3.5)$$

Remark 3.2. If we set $\gamma = \sigma = c = d = 1$ in (3.4), we obtained the known result of (see [9])

$$B\, L_{\alpha,\beta}^{(p,\mu,v)}(x+1,y) + B\, L_{\alpha,\beta}^{(p,\mu,v,)}(x,y+1) = B\, L_{\alpha,\beta}^{(p,\mu,v,)}(x,y) \qquad (3.6)$$

Remark 3.3. If we set $c = d = 1$ in (3.4), we obtained the following new result

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x+1,y) + B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y+1) = B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y) \qquad (3.7)$$

Theorem 3.3. For any $\alpha, \beta, c, d, x, y > 0$, the following assertions holds true:

$$\min(c,d)\, B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y) \leq B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) \leq \max(c,d)\, B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y). \qquad (3.8)$$

Proof: From the inequality,

$$min(c,d) \leq \sqrt{cd} \leq L(c,d) \leq \left(\frac{c+d}{2}\right) \leq max(c,d)$$

$$\text{and} \quad B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)} > 0,$$

we get the following relation

$$\min(c,d)\, B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y) \leq B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) \qquad (3.9)$$

Using the following well known young's inequality

$$c^{1-t}\, d^t \leq c(1-t) + dt \qquad \forall\, t \in (0,1)$$

we obtain

$$B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) \leq cB_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y+1) + dB_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x+1,y) \leq max(c,d)\left(B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y+1) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x+1,y)\right).$$

Using the relation (3.6), we achieved the desired result.

Theorem 3.4. For any $\alpha, \beta, c, d, x, y > 0$, the following assertions holds true:

$$B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+n,y+1) \tag{3.10}$$

Proof: Using the series representation $(1-t)^{-1} = \sum_{n=0}^{\infty} t^n$, for $t \in (0,1)$ with the arguments of uniform convergence of this power series, we have

$$B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = \sum_{n=0}^{\infty}\int_0^1 c^{1-t}\, d^t\, t^{x+n-1}(1-t)^y \, E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)dt. \tag{3.11}$$

Using the definition (2.1) in the above expression, we achieved the desired result.

Theorem 3.5. Let $\alpha,\beta,c,d,x,y>0$, the following representation holds true:

$$B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = \sum_{m,n=0}^{\infty}\frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x+n,y+m)}{m!\,n!}(\log(c))^m(\log(d))^n. \tag{3.12}$$

Proof: The following power series expansion

$$c^{1-t}\sum_{m=0}^{\infty}\frac{(\log(c))^m}{m!}(1-t)^m \quad , \quad d^t\sum_{n=0}^{\infty}\frac{(\log(d))^n}{n!}t^n$$

using the above expansion in the result (2.1), we have

$$B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = \int_0^1\sum_{m,n=0}^{\infty}\frac{(\log(c))^m(\log(d))^n}{m!\,n!}t^{x+n-1}(1-t)^{y+m-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)dt.$$

Combined with the fact that the entire series converges uniformly, we can replace the order of the integral with the resulting infinite sum (3.12).

Theorem 3.6. For any $c,d,\mu,v,\gamma,\sigma>0$ a, the following relation holds true:

$$B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) = 2\int_0^{\frac{\pi}{2}}\left(\frac{c}{d}\right)^{\cos^2\theta}\cos^{2x-1}\theta\,\sin^{2y-1}\theta \times E_{\alpha,\beta}^{\gamma,\sigma}(-p(\sec^2\theta)^{\mu}(cosec^2\,\theta)^{v})d\theta, \tag{3.13}$$

Proof: Let $t=\cos^2\theta$ in (2.1). After simplification, we obtain the desired result (3.13).

4. The random variable of logarithmic beta function $BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)$

We will now describe the beta logarithmic distribution for (2.1) and calculate its average, spread, and ability to generate moments.

For $\alpha,\beta,c,d,x,y>0$, the beta logarithmic distribution is defined as

$$f(t) = \begin{cases}\dfrac{1}{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y)}c^{1-t}d^t\,t^{x-1}\,(1-t)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) & (0<t<1)\\ 0, \;\ldots\ldots\ldots\ldots & otherwise\end{cases} \tag{4.1}$$

$$c,d,\alpha,\beta,\gamma,\sigma\in\mathbb{R}^+, \quad p\ge 0.$$

We have the k^{th} moment of a random variable X defined as

$$\rho = \mathbb{E}(X^k) = \frac{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d\,;x+d,\ y)}{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)}, \tag{4.2}$$

When $k=1$, the mean is obtained as a special case of (4.2) given by

$$\rho = \mathbb{E}(X) = \frac{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d\,;x+1,\ y)}{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)}, \tag{4.3}$$

The variance of a distribution is discuss as follows:

$$\sigma^2 = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$$

$$= \frac{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y) + BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+2,y)}{\left\{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)\right\}^2} - \frac{\left\{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+1,y)\right\}^2}{\left\{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)\right\}^2} \tag{4.4}$$

The distribution's moment generating function (mgf) is defined as

$$M(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbb{E}(X^n) = \frac{1}{B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)} \times \sum_{n=0}^{\infty} B\,L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+n,y)\frac{t^n}{n!}. \tag{4.5}$$

In this case, a well-known lemma comes to mind.

Lemma 4.1. Let $\mathcal{Y}$ be a random variable with values that fall within a certain range [c, d].

Then we have for all $\epsilon \in [a,b]$.

$$\left|P(\mathcal{Y} \le \epsilon) - \frac{d-E(Y)}{d-c}\right| \le \frac{1}{2} + \frac{\left|\epsilon - \frac{c+d}{2}\right|}{d-c} \quad (4.6)$$

Proposition 4.2. Assume X is a beta logarithmic random variable with parameters

$(c,d;x,y)$. The following assumptions are true any $k, \epsilon > 0$:

$$\left|P(X \le \epsilon) - \frac{B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y+1)}{B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)}\right| \le \frac{1}{2} = \left|\epsilon - \frac{1}{2}\right| \quad (4.7)$$

and

$$P(X^k \ge \epsilon) - \frac{B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+k,y)}{\epsilon\, B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)} \quad (4.8)$$

proof: With the help of (3.4) and (4.3), we have

$$\rho = \mathbb{E}(X) = 1 - \frac{B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d\,;x,\ y+1)}{B\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x,y)}, \quad (4.9)$$

we obtained the desired result (4.6) by using the above relation in the inequality (4.7).

The Markov's inequality can be used to deduce the second inequality (4.8).

$$P(X^k \ge \epsilon) \le \frac{\mathbb{E}(X^k)}{\epsilon}$$

and the definition of $\mathbb{E}(X^k)$, we get the desired result (4.8).

5. Hypergeometric and Confluent hypergeometric functions in terms of extended Logarithmic mean

In this section, we defined hypergeometric and confluent hypergeometric functions in terms of Extended beta logarithmic functions

The extension of hypergeometric logarithmic function is defined as follows:

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;\omega;z) = \sum_{n=0}^{\infty}(x)_n \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;\ y+n,\omega-y)}{B(y,\omega-y)}\frac{Z^n}{n!}. \quad (5.1)$$

$$p \ge 0;\ \ |z|<1\,;\ \ \alpha,\beta,\mu,v,\gamma,\sigma > 0,\quad \Re(\omega) > \Re(y) > 0,\quad c,d > 0.$$

The extension of confluent hypergeometric logarithmic function is defined as follows:

$$\Phi\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(y;\omega;z) = \sum_{n=0}^{\infty}(x)_n \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;y+n,\omega-y)}{B(y,\omega-y)}\frac{Z^n}{n!}. \quad (5.2)$$

$$p \ge 0;\quad c,d,\alpha,\beta,\mu,v,\gamma,\sigma > 0,\quad \Re(\omega) > \Re(y) > 0.$$

Remark 5.1. If we choose $c = d = 1$ in (5.1) and (2.2), then we achieved the known result given by (see [9]).

Remark 5.2. If we choose $\gamma = \sigma = 1$ in (5.1) and (2.2), then we achieved the known result given by (see[10]).

5.1. Representation of the integral

Theorem 5.1. The hypergeometric and confluent hypergeometric logarithmic functions have the following integral representations:

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y;\omega;z) = \frac{1}{B(y,\omega-y)}\int_0^1 c^{1-t}d^t t^{y-1} \times (1-t)^{\omega-y-1}(1-zt)^{-x} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{\mu}(1-t)^{v}}\right)dt, \quad (5.3)$$

$$(p \ge 0;\ \ c,d,\alpha,\beta,\mu,v,\gamma,\sigma \in \mathbb{R}^+;\ and\ \ |arg(1-z)| < \pi\,;\ \ \Re(\omega) > \Re(y) > 0),$$

and

$$\Phi\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(y;\omega;z) = \frac{1}{B(y,\omega-y)} \times \int_0^1 c^{1-t}d^t t^{y-1}(1-t)^{\omega-y-1}e^{zt}\ E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{\mu}(1-t)^{v}}\right)dt, \quad (5.4)$$

$$(p \ge 0;\ \ c,d,\alpha,\beta,\mu,v,\gamma,\sigma \in \mathbb{R}^+;\quad \Re(\omega) > \Re(y) > 0).$$

Proof: If we rearranging integration and summation, after using the definition (2.1), we get

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y;\omega;z) = \frac{1}{B(y,\omega-y)} \times \int_0^1 c^{1-t}d^t t^{y-1}(1-t)^{\omega-y-1}(1-zt)^{-x} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{\mu}(1-t)^{v}}\right)\sum_{n=0}^{\infty}(x)_n\frac{(zt)^n}{n!}\,dt, \quad (5.5)$$

We obtained the desired result (4.3) by using the binomial theorem in (5.5).

Similarly, (5.4) can be obtained.

5.2. Formula of derivative

Theorem 5.2. The following derivative formulae are right:

$$D_z^n\left\{F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y;\omega;z)\right\} = \frac{(x)_n(y)_n}{(\omega)_n} \times F\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;x+n,y+n;\omega+n;z) \quad (5.6)$$

and

$$D_z^n\left\{\Phi\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(y;\omega;z)\right\} = \frac{(y)_n}{(\omega)_n}\Phi\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;y+n;\omega+n;z), \quad (5.7)$$

where

$$(p \geq 0;\ \ c, d, \alpha, \beta, \mu, v, \gamma, \sigma \in \mathbb{R}^{+};\ \ \Re(\omega) > \Re(y) > 0\ \ ;\ \ n > \mathbb{N}_0).$$

Proof: We are aware of the well-known relationship between Euler beta functions.

$$B(y, \omega - y) = \frac{\omega}{y} B(y + 1, \omega - y). \tag{5.8}$$

We get by differentiating (5.1) with respect to variable z

$$D_z F\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c, d; \omega; z) = \sum_{n=0}^{\infty} (x)_n \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c, d; y + n, \omega - y)}{B(y, \omega - y)} \frac{Z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (x)_{n+1} \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c, d; y + n + 1, \omega - y)}{B(y, \omega - y)} \frac{Z^n}{n!}, \tag{5.9}$$

using (1.4) and (5.8) in the above expression, we obtain

$$D_z F\, L_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c, d; \omega; z) = \frac{xy}{\omega} \sum_{n=0}^{\infty} (x+1)_n \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c, d; y + n + 1, \omega - y)}{B(y + 1, \omega - y)} \frac{Z^n}{n!} \tag{5.10}$$

To achieve the required result, repeat the process up to (n-1).

Similarly, applying the same process to 5.2 yields the desired result (5.7).

Remark 5.3. We obtain the known result proposed by (see [10]), if we use $\gamma = \sigma = 1$ in the expressions (5.6) and (5.7).

Remark 5.4. Choosing c=d=γ=σ=1 in expressions (5.6) and (5.7) yields the well-known results introduced by Khan et al. (see [9]).

6. Formulas for transformation

Theorem 6.1. The hypergeometric logarithmic and confluent hypergeometric logarithmic functions have the following formulas:

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, y; \omega; z) = (1 - z)^{-x} F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(x, \omega - y; \omega; -\frac{z}{1-z}\right), \tag{6.1}$$

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(x, y; \omega; 1 - \frac{1}{z}\right) = z^x F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, \omega - y; \omega; 1 - z), \tag{6.2}$$

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(x, y; \omega; \frac{z}{1-z}\right) = (1 + z)^x F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, \omega - y; \omega; -z), \tag{6.3}$$

$$\Phi\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(y; \omega; z) = e^z L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\omega - y; \omega; -z), \tag{6.4}$$

$$(p \geq 0;\ \ c, d, \alpha, \beta, \mu, v, \gamma, \sigma \in \mathbb{R}^{+};\ \ \ \Re(\omega) > \Re(y) > 0).$$

Proof: Replacing t by $1 - t$ in $(1 - zt^{-x})$ and substituting

$$\left(1 - z(1 - t)\right)^{-x} = (1 - z)^{-x}\left(1 + \frac{z}{1 - z} t\right)^{-x}$$

in (5.3), we obtain

$$F\, L_{a,b,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, y; \omega; z) = \frac{(1 - z)^{-x}}{B(y, \omega - y)} \times \int_0^1 c^{1-t} d^t t^{y-1} (1 - t)^{\omega-y-1} \left(1 + \frac{z}{1 - z} t\right)^{-x} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{\mu}(1-t)^{v}}\right) dt, \tag{6.5}$$

further, we have

$$F\, L_{a,b,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, y; \omega; z) = \frac{(1 - z)^{-x}}{B(y, \omega - y)} \times \int_0^1 c^{1-t} d^t t^{y-1} (1 - t)^{\omega-y-1} \left(1 - \frac{-z}{1 - z} t\right)^{-x} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{\mu}(1-t)^{v}}\right) dt, \tag{6.6}$$

In light of (5.3), we obtain the desired outcome (6.1).

Replacing z by $1 - \frac{1}{z}$ and $\frac{1}{1+z}$ in (6.1) gives (6.2) and (6.3) respectively.

Applying the same process as in (6.1), we can obtain (6.4) through simple calculation.

Theorem 6.2. The following relation holds true:

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, y; \omega; 1) = \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c, d; x, \omega - x - y)}{B(y, \omega - y)}, \tag{6.7}$$

$$(p \geq 0;\ \ c, d, \alpha, \beta, \mu, v, \gamma, \sigma \in \mathbb{R}^{+};\ \ \ \Re(\omega - x - y) > 0).$$

Proof: We get the desired result (6.7) by putting $\mathbf{z = 1}$ in (5.3) and using the definition (2.1).

7. Generating function

Theorem 7.1. The following is the relationship between the generating function for

$$F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x, y; \omega; z)$$

$$\sum_{k=0}^{\infty} (x)_k F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x + k, y; \omega; z) \frac{t^k}{k!} = (1 - z)^{-x} F\, L_{c,d,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(x, y; \omega; \frac{z}{1-z}\right) \tag{7.1}$$

Proof: Let τ be the left hand side of (7.1), then from (5.1), we have

$$\tau = \sum_{n=0}^{\infty} (x)_k \left(\sum_{n=0}^{\infty} \frac{(x+k)_n BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;y+n,\omega-y)}{B(y,\omega-y)} \frac{Z^n}{n!} \right) \frac{t^k}{k!}, \tag{7.2}$$

Using the well-known identity

$(a)_n(a+n)_k = (a)_k(a+k)_n$, we obtain

$$\tau = \sum_{n=0}^{\infty} (x)_k \frac{(x+k)_n BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;y+n,\omega-y)}{B(y,\omega-y)} \left(\sum_{k=0}^{\infty} (x+n)_k \frac{t^k}{k!} \right) \frac{Z^n}{n!}, \tag{7.3}$$

Since we know that $\sum_{k=0}^{\infty}(x+n)_k \frac{t^k}{k!} = (1-t)^{-x-k}$, we have

$$\tau = (1-t)^{-x} \sum_{n=0}^{\infty} (x)_k \frac{BL_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(c,d;y+n,\omega-y)}{B(y,\omega-y)} \left(\frac{z}{1-z}\right)^n \frac{1}{n!} \tag{7.4}$$

Finally, we get the right side of (7.1) by using (5.1) in (7.4).

8. Conclusion

This article discusses the statistical features of an extension of expanded beta functions using the logarithmic mean, as well as the algebraic properties of this type of extension and its application in probability. This logarithmic mean based extension of expanded beta functions has its roots in science and engineering. There are also many future applications for this type of function, such as Apple functions, the Reimann-Liouville fractional derivative operator, and the Whittaker functions, all with the goal of improving their properties.

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مقالة بحثية

امتداد جديد لدالة بيتا اللوغاريتمية وخصائصها

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المُلخّص

قدم خان وآخرون، نوعًا جديدًا من دالة بيتاء اللوغاريتمية ونهدف في هذه المقالة البحثية إلى تقديم امتداد جديد لوظيفة بيتا اللوغاريتمية، علاوة على ذلك، قمنا بدراسة خصائصها الأساسية ومناقشة الصيغ المتنوعة لهذا الامتداد مثل التمثيل التكاملي، صيغة الجمع، صيغة التحويل وخصائصها الإحصائية. بناءً على هذا المفهوم، قمنا بتقديم دوال جديدة فوق هندسية ودوال الكونفولونت فوق الهندسية ودراسة خصائصها.

الكلمات المفتاحية: المتوسط اللوغاريتمي، دالة بيتا، دالة فوق هندسية، دالة الكونفولونت فوق هندسية.

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