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RESEARCH ARTICLE

ON GENERALIZED – $K_{(B)}$ QUAD – RECURRENT IN FINSLER SPACE

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Abstract

In this paper, we study the generalized $K_{(B)}$-quad recurrent Finsler spaces of fourth order with a non-metric Finsler connection. We obtain the expressions of the fourth order recurrent curvature tensor and the h-fourth order recurrent curvature tensor. We also study the properties of these tensors and obtain some interesting results.

Keywords: Recurrent Finsler space, Fourth order recurrent Finsler space, Non-metric Finsler connection, Curvature tensor.

1. Introduction

Finsler spaces are a generalization of Riemannian spaces. They were introduced by Finsler in 1924. A Finsler space is a manifold M equipped with a Finsler metric F, which assigns to each point $p \in$ M and each non-zero tangent vector $x \in T_{pM}$ a non-negative real number $F(p, x)$. The Finsler metric F is a generalization of the Riemannian metric. In a Riemannian space, the metric is given by a Riemannian tensor G, which assigns to each point $p \in$ M a symmetric, positive definite bilinear form $G_p: T_P M \times T_P M \to R$. Recurrent Finsler spaces were introduced by Chern in 1936. A Finsler space F is said to be recurrent if there exists a non-zero vector field V such that the following condition holds: $\nabla_{XV} = kV$, where ∇ is the Finsler connection, k is a scalar function, and V is a vector field. Second order recurrent Finsler spaces were introduced by B. S. Yadav and U. C. Pandey in 1984. A Finsler space F is said to be second order recurrent if there exists a non-zero vector field V such that the following condition holds: $\nabla_{XV} = k_1 V + k_2 H$, where ∇ is the Finsler connection, k_1 and k_2 are scalar functions, V is a vector field, and H is the h-vector field. Generalized recurrent and birecurrent affinely connected spaces have been studied by a number of mathematicians (see [2], [3], [4], [5], [10], [11], [12] and [13]) and there is a rich body of literature on this topic. Some of the key researchers in this area include (see [1], [6], [7] and [13]):

Berwald [8] defined covariant derivative for his connection parameter G^i_{jk}. Thus Berwald's covariant derivative $\mathcal{B}_k T^i_j$ *or* $T^i_{j(\mathcal{B})}$ of an arbitrary tensor field T^i_j with respect to x^k is given by [14].

(1.1) $\quad T^i_{j(B)} = \partial_k T^i_j - (\dot\partial_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}$.

Berwald's covariant derivative of the metric function F, the unit vector ℓ^i and vector $\dot x^i$ vanish identically, i.e.

(1.2) (a) $\quad F_{(k)} = 0$, (b) $\ell^i_{(k)} = 0$ and (c) $\dot x^i_{(k)} = 0$.

In general, Berwald's covariant derivative of the metric tensor g_{ij} doesn't vanish and is given by (see [9])

(1.3) $\quad g_{ij(k)} = -2C_{ijk|h}\dot x^h = -2\dot x^h C_{ijk(h)}$.

The tensor C_{ijk} is called (h) hv-torsion tensor, defined by [14]

(1.4) $\quad C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij} = \frac{1}{4}\dot\partial_k \dot\partial_i \dot\partial_j F^2$.

The tensor C^i_{jk} is the associate tensor of C_{ijk} defined by

(1.5) (a) $\quad C_{jsk} = C^i_{jk} g_{is}$ and (b) $C_{jsk}\, g^{ji} = C^i_{sk}$.

This tensors satisfy the following identities

(1.6) (a) $\quad C_{ijk}\dot x^i = C_{jki}\dot x^i = C_{kij}\,\dot x^i = 0$,

(b) $C^i_{jk}\,\dot x^k = C^i_{kj}\,\dot x^k = 0$, (c) $C^i_{jk}\dot x_i = 0$,

(d) $C^r_{jr} = C_j$ and (e) $C^i_{jk}\, g^{jk} = C^i$.

The vectors $\dot x_i, \dot x^i$ and the metric tensor $g_{ij}\,(x, \dot x)$ satisfy

(1.7) (a) $g_{ij}\, g^{ik} = \delta_j^k = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$,

(b) $\delta_h^i\, g_{ik} = g_{hk}$, (c) $g_{ik}\, g^{ik} = \delta_k^k = n$,
(d) $\delta_h^i\, g^{hk} = g^{ik}$, (e) $\dot{x}_i \dot{x}^i = F^2$,

(h) $g_{ij} = \dot{\partial}_i \dot{x}_j = \dot{\partial}_j \dot{x}_i$ and
(i) $\dot{x}_i = g_{ij}\,(x, \dot{x})\, \dot{x}^j$.

The tensor K_{jkh}^i is called Cartan's fourth curvature tensor, it is positively homogeneous of degree zero in $\dot{x}^i$, which defined by

$$K_{jkh}^i = \partial_h \Gamma_{kj}^{*i} + (\dot{\partial}_s\, \Gamma_{jh}^{*i}) G_k^s + \Gamma_{th}^{*i}\, \Gamma_{kj}^{*t} - h/k \;.$$

The curvature tensor K_{jkh}^i is skew-symmetric in its last two lower indices, i.e.

(1.8) $K_{jkh}^i = -\,K_{jhk}^i$.

The associate curvature tensor K_{ijkh} of the curvature tensor K_{jkh}^i is given by

(1.9) $K_{ijkh} = g_{rj}\; K_{ikh}^r$.

The tensor K_{jkh}^i and its associate curvature tensor K_{ijkh} satisfy the following relations:

(1.10) (a) $K_{ijkh} + K_{ijhk} = -2C_{ijs} H_{hk}^s$,

(b) $K_{jikh} + K_{jkih} + K_{jhki} + 2\dot{x}^r \big(C_{jis} K_{rhk}^s + C_{jks} K_{rih}^s + C_{jhs} K_{rki}^s\big) = 0$,

(c) $K_{jkh}^i + K_{hjk}^i + K_{khj}^i = 0$ and

(d) $K_{jikh} + K_{hijk} + K_{kihj} = 0$.

The Ricci tensor K_{jk} , the curvature vector K_j and the curvature scalar K are connected by

(1.11) (a) $K_{jki}^i = K_{jk}$, (b) $K_{jk}\, \dot{x}^k = K_j$

and (c) $K_{jk}\, g^{jk} = k$.

Berwald curvature tensors of H_{jkh}^i , the h(v)-torsion tensor H_{kh}^i and Cartan's fourth curvature tensor K_{jkh}^i are connected by

(1.12) (a) $H_{jkh}^i = K_{jkh}^i + \dot{x}^m (\dot{\partial}_j K_{mkh}^i)$

and (b) $K_{jkh}^i\, \dot{x}^j = H_{kh}^i$,

where the tensor H_{jkh}^i is called h-curvature tensor of Berwald and the tensor H_{kh}^i is called h(v)- torsion tensor and defined as

(1.13) (a) $H_{jkh}^i = \partial_h G_{jk}^i + G_{jk}^r G_{rh}^i + G_{rjh}^i G_k^r - h/k^*$

and (b) $H_{kh}^i = \partial_h G_k^i + G_k^r G_{rh}^i - h/k$.

The curvature tensor of Berwald H_{jkh}^i and the h(v)-torsion tensor H_{kh}^i are related

(1.14) (a) $\partial_r H_{kh}^i = H_{rkh}^i$ and (b) $H_{jkh}^i \dot{x}^j = H_{kh}^i$.

The tensor H_k^i called the deviation tensor, given by

(1.15) (a) $H_h^i = 2\partial_h G^i - \partial_s\, G_k^i \dot{x}^s + 2G_{hs}^i G^s - G_s^i G_h^s$ and

(b) $H_{kh}^i \dot{x}^k = -\,H_{hk}^i \dot{x}^k = H_h^i$.

H-Ricci tensor H_{jk} , curvature vector H_k and the curvature scalar H are connected by

(1.16) (a) $H_{jkr}^r = H_{jk}$, (b) $H_{kr}^r = H_k$
and

(c) $H = \frac{1}{(n-1)}\, H_r^r$.

The above tensors also satisfy the following:

(1.17) (a) $H_{kh(m)}^i + H_{mk(h)}^i + H_{hm(k)}^i = 0$,
(b) $H_{kh}^i = \dot{\partial}_k H_h^i$, (c) $H_{kh}\, \dot{x}^k = H_h$,
(d) $H_k\, \dot{x}^k = (n-1)H$, (f) $\dot{x}_i\, H_j^i = 0$,
(g) $g_{ij} H_k^i = g_{ik} H_j^i$ and (h) $H_{hk} - H_{kh} = H_{ikh}^i$.

2. A Generalized – $K_{(B)}$ Quad – Recurrent in Finsler space

Let us consider a Finsler space F_n for which Cartan's fourth curvature tensor K_{jkh}^i satisfies the generalized recurrence property with respect to Berwald's connection parameter G_{kh}^i, i.e. characterized by the following condition:

(2.1) $K_{jkh(l)(m)(n)(s)}^i = a_{lmns}\, K_{jkh}^i + b_{lmns}\big(\delta_h^i g_{jk} - \delta_k^i g_{jh}\big) - 2\dot{x}^r d_{lmn}\big(\delta_h^i C_{jks} - \delta_k^i C_{jhs}\big)_{(r)}$
$- 2\dot{x}^q \theta_{lms}\big(\delta_h^i C_{jkn} - \delta_k^i C_{jhn}\big)_{(q)}$
$- 2\dot{x}^q b_{lm}\big(\delta_h^i C_{jkn} - \delta_k^i C_{jhn}\big)_{(q)(s)}$
$- 2\dot{x}^p \eta_{lns}\big(\delta_h^i C_{jkm} - \delta_k^i C_{jhm}\big)_{(p)}$
$- 2\dot{x}^p \alpha_{ln}\big(\delta_h^i C_{jkm} - \delta_k^i C_{jhm}\big)_{(p)(s)}$
$- 2\dot{x}^p \varphi_{ls}\big(\delta_h^i C_{jkm} - \delta_k^i C_{jhm}\big)_{(p)(n)}$
$- 2\dot{x}^p \mu_l\big(\delta_h^i C_{jkm} - \delta_k^i C_{jhm}\big)_{(p)(n)(s)}$,

where $a_{lmns} = c_{lmn(s)} + c_{lmn}\,\lambda_s$ and $b_{lmns} = c_{lmn} u_s + d_{lmn(s)}$ are non-zero covariant tensor field of fourth order, $\theta_{lms} = b_{lm(s)}$ and $\eta_{lns} = \alpha_{ln(s)}$ are non–zero covariant tensors field of third order, $\varphi_{ls} = \mu_{l(s)}$ is non–zero covariant tensor field of second order, $(l)(m)(n)$(s) is Berwald's covariant differential operator with respect to x^l, x^m, x^n and x^s, respectively, is covariant derivative of fourth order in sense of Berwald.

Definition 2.1. A Finsler space F_n for which Cartan's fourth curvature tensor K_{jkh}^i satisfies the condition (2.1) will be called a generalized $K_{(\mathcal{B})}$- quad recurrent space, where a_{lmns} and b_{lmns} are non-zero vectors field and the tensor will be called generalized $\mathcal{B}$ – quad- recurrent tensor. We shall denote such space and tensor briefly as $GK_{(\mathcal{B})}$- QRF_n and $G\mathcal{B} - QR$, respectively.

Lemma 2.1. Every generalized $K_{(\mathcal{B})}$-recurrent in Finsler space is generalized $K_{(\mathcal{B})}$ -quad recurrent in Finsler space.

Contracting the indices i and h in (2.1), using (1.7c) and (1.11a), we get

(2.2) $K_{jk(l)(m)(n)(s)} = a_{lmns}\, K_{jk} + (n-1)\{\, b_{lmns}\, g_{jk} - 2\dot{x}^r d_{lmns}\, C_{jks(r)} - 2\dot{x}^q \theta_{lms}\, C_{jkn(q)} - 2\dot{x}^q b_{lm}\, C_{jkn(q)(s)} - 2\dot{x}^p \eta_{lns} C_{jkm(p)} - 2\dot{x}^p \alpha_{ln}\, C_{jkm(p)(s)} - 2\dot{x}^p \varphi_{ls}\, C_{jkm(p)(n)} - 2\dot{x}^p u_l\, C_{jkm(n)(p)(s)}\}$.

Theorem 2.1. In $GK_{(B)}$- QRF_n , Berwald's covariant derivative of the fourth order for the K−Ricci tensor K_{jk} is given by (2.2).

Transvecting the condition (2.2) by $\dot{x}^k$, using (1.2c), (1.11b) and (1.6a), we get

(2.3) $K_{j(l)(m)(n)(s)} = a_{lmns}\, K_j + (n-1)\, b_{lmns}\, \dot{x}_j$.

The equation (2.3), shows that, the curvature vector K_j can't vanish, because the vanishing to would imply $b_{lmns} = 0$, a contradiction.

Theorem 2.2. In $GK_{(B)}$- QRF_n , the curvature vector K_j is non-vanishing.

Transvecting condition (2.1) by $\dot{x}^j$, using (1.2c), (1.12b), (1.7i) and (1.6a), we get

(2.4) $H^i_{kh(l)(m)(n)(s)} = a_{lmns} H^i_{kh} + b_{lmns}(\delta^i_h \dot{x}_k - \delta^i_k \dot{x}_h)$.

Transvecting (2.4) by $\dot{x}^k$, using (1.2c), (1.15b), (1.7d) and (1.7e), we get

(2.5) $H^i_{h(l)(m)(n)(s)} = a_{lmns} H^i_h + b_{lmns}(\delta^i_h F^2 - \dot{x}^i \dot{x}_h)$.

Thus, we conclude

Theorem 2.3. In $GK_{(B)}$- QRF_n , Berwald's covariant derivative of the fourth order for the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h is given by (2.4) and (2.5), respectively.

Contracting the indices i and h in the equations (2.4) and (2.5), using (1.16b), (1.16c), (1.7i) and (1.7c), we get

(2.6) $H_{k(l)(m)(n)(s)} = a_{lmns} H_k + (n-1)\, b_{lmns} \dot{x}_k$.

(2.7) $H_{(l)(m)(n)(s)} = a_{lmns}\, H + (n-1) b_{lmns} F^2$.

The equations (2.6) and (2.7), show that, the curvature vector H_k and the curvature scalar H can't vanish, because the vanishing of any one of them would imply

$b_{lmns} = 0$, a contradiction.

Theorem 2.4. In $GK_{(B)}$- QRF_n , the curvature vector H_k and the curvature scalar H are non-vanishing.

Interchanging the indices n and r in equation (2.4) and subtracting the equation obtained from (2.4), we get

(2.8) $H^i_{kh(l)(m)(n)(s)} - H^i_{kh(l)(m)(s)(n)} = (a_{lmns} - a_{lmsn}) + (b_{lmns} - b_{lmsn})(\delta^i_h \dot{x}_k - \delta^i_k \dot{x}_h)$.

If the recurrence covariant tensor field of fourth order b_{lmsn} is symmetric in third and fourth indicator, then above equation can by writing as

(2.9) $H^i_{kh(l)(m)(n)(s)} - H^i_{kh(l)(m)(s)(n)} = (a_{lmns} - a_{lmsn}) H^i_{kh}$.

Theorem 2.5. In $GK_{(B)}$- QRF_n , if the recurrence covariant tenser field of fourth order b_{lmsn} is symmetric, the commutation formula Berwald's covariant derivatiation is given by identity (2.9).

Adding the expression obtained cyclic change of (2.9), with respect to the indices k, h and l and using (1.17.a), we get

$$\left(H^i_{kh(l)(m)(n)(s)} - H^i_{kh(l)(m)(s)(n)}\right) + \left(H^i_{hl(k)(m)(n)(s)} - H^i_{hl(k)(m)(s)(n)}\right) + \left(H^i_{lk(h)(m)(n)(s)} - H^i_{lk(h)(m)(s)(n)}\right) = 0 ,$$

using (2.9), in above equation, we get

(2.10) $(a_{lmns} - a_{lmsn}) H^i_{kh} + (a_{kmns} - a_{kmsn}) H^i_{hl} + (a_{hmns} - a_{hmsn}) H^i_{lk} = 0$.

Corollary (2.1). In $GK_{(B)}$- QRF_n , if the recurrence covariant tensor field of fourth order b_{lmsn} is symmetric, the h(v)-torsion H^i_{kh} satisfies identity (2.10).

By cyclic rotation of the indices k , h , l and m in (2.10), we get

(2.11) $(a_{hlns} - a_{hlsn}) H^i_{mk} + (a_{mlns} - a_{mlsn}) H^i_{kh} + (a_{klns} - a_{klsn}) H^i_{hm} = 0$,

(2.12) $(a_{khns} - a_{khsn}) H^i_{lm} + (a_{lhns} - a_{lhsn}) H^i_{mk} + (a_{mhns} - a_{mhsn}) H^i_{kl} = 0$

and

(2.13) $(a_{mkns} - a_{mksn}) H^i_{hl} + (a_{hkns} - a_{hksn}) H^i_{lm} + (a_{lkns} - a_{lksn}) H^i_{mh} = 0$.

By using the skew-symmetric property of the h(v)-torsion tensor H^i_{lm} in its lower indices and adding (2.10), (2.11), (2.12) and (2.13), we get

(2.14) $[(a_{lmns} - a_{lmsn}) + (a_{mlns} - a_{mlsn})]\, H^i_{kh} + [(a_{hlns} - a_{hlsn}) + (a_{lhns} - a_{lhsn})]\, H^i_{mk} + [(a_{khns} - a_{khsn}) + (a_{hkns} - a_{hksn})]\, H^i_{lm} + [(a_{mkns} - a_{mksn}) + (a_{kmns} - a_{kmsn})] H^i_{hl} + [(a_{hmns} - a_{hmsn}) - (a_{mhns} - a_{mhsn})] H^i_{lk} + [(a_{klns} - a_{klsn}) - (a_{lkns} - a_{lksn})] H^i_{hm} = 0$.

If the recurrence covariant tensor a_{lmns} is symmetric in first and second indicator, then above equation can be written as

$$(2.15)\quad (a_{lmns} - a_{lmsn})\, H^i_{kh} + (a_{mkns} - a_{mksn})\, H^i_{hl} + (a_{khns} - a_{khsn})\, H^i_{lm} + (a_{hlns} - a_{hlsn})\, H^i_{mk} = 0 .$$

Theorem 2.6. In $GK_{(B)}$-QRF_n , if the recurrence covariant tenser field a_{lmns} is symmetric, than the h(v)-torsion tensor H^i_{kh} satisfies the identity (2.15).

If the recurrence covariant tensor a_{lmns} is skew-symmetric in first and second indicator, then equation (2.15) can be written as

(2.16) $(a_{hmns} - a_{hmsn})\, H^i_{lk} + (a_{klns} - a_{klsn})\, H^i_{hm} = 0$.

Corollary (2.2). In $GK_{(B)}$- QRF_n , if the recurrence covariant tensor field a_{lmns} is skew-symmetric, then the h(v)-torsion tensor H^i_{kh} satisfies the identity (2.16).

If the recurrence covariant tensor field of fourth order b_{lmsn} is skew-symmetric in third and fourth indicator, equation (2.8) can by written as

(2.17) $H^i_{kh(l)(m)(n)(s)} - H^i_{kh(l)(m)(s)(n)}$

$$= (a_{lmns} - a_{lmsn})\, H^i_{kh} + 2b_{lmns}\left(\delta^i_h \dot{x}_k - \delta^i_k \dot{x}_h\right) .$$

Corollary (2.3). In $GK_{(B)}$- QRF_n , if the recurrence covariant tensor field of fourth order is skew-symmetric, the commutation formula Berwald's s covariant derivative is given by (2.17).

Contracting the indices i and l in (2.10), using (1.16b) and (1.15b), we get

$$(a_{imns} - a_{imsn})H^i_{kh} + (a_{kmns} - a_{kmsn})H_h - (a_{hmns} - a_{hmsn})H_k = 0 ,$$

above equation can be written as

(2.18) $$H^i_{kh} = \frac{1}{(a_{imns} - a_{imsn})}[(a_{hmns} - a_{hmsn})H_k - (a_{kmns} - a_{kmsn})H_h] .$$

Theorem 2.7. In $GK_{(B)}$- QRF_n , if the recurrence covariant tensor field of fourth order b_{lmsn} is symmetric, the h(v)-torsion H^i_{kh} defined by the formula (2.18).

3. A Generalized $K_{(B)} - QR -$ Affinely Connected Space

Affine connection spaces, introduced by Élie Cartan, are fundamental in differential geometry. A parallel transport mechanism allows for vector comparison along smooth curves. This concept is essential in general relativity and theoretical physics. Generalized affine connected spaces, also known as Finsler spaces with a linear connection, extend the notion of affine connection spaces by incorporating a Finsler metric. This metric introduces a non-Euclidean measure of distance between points, generalizing the standard Euclidean distance employed in affine connection spaces. The interplay between the Finsler metric and the linear connection in these spaces gives rise to intriguing geometric and dynamical properties.

In this research, we explore the intricate world of generalized affine connected spaces. We aim to unravel their unique characteristics and uncover the profound connections they hold with other mathematical frameworks.

Definition (3.1). A finsler space F_n whose connected parameter G^i_{jk} is independent of $\dot{X}^i$ is called an affinely connected space (Berwald space), thus an affinely connected space is characterized by any of the following equivalent equations:

(3.1) $G^i_{jkh} = 0$ and (3.2) $C_{ijk|h} = 0$.

Remark (3.1). The connection parameters of Cartan and G^i_{jk} of Berward coincide in affinely connected space and they are independent of directional argument i.e.

(3.3) (a) $\dot{\partial}_j G^i_{kh} = 0$ and (b) $\dot{\partial}_j G\Gamma^i_{kh} = 0$.

Remark (3.2). In an affinely connected space Berwald's converiant derivative of the metric tensor g_{ij} and its associate g^{ij} are vanish [R], i.e.

(3.4) (a) $g_{ij(l)} = 0$ and (b) $g^{ij}_{(l)} = 0$.

In view of (3.4a) and (1.3), we get

(3.5) $\dot{X}^l C_{ijk(l)} = 0$.

Definition (3.2). The generalized $K_{(B)}$-quad-recurrent space which is an affinely connected space [satisfies any of the conditions (3.1), (3.2), (3.3a) and (3.3b)] will be called a generalized $K_{(B)}$-quad-recurrent affinity connected space and will be denote it briefly by $GK_{(B)}$- QR - affinity connected space.

Remark (3.3). It will be sufficient to call the tensor which satisfies the condition of $GK_{(B)}$- QR- affinity connected space as generalized $B - QR$ tensor (briefly $GB - QR$).

Let us consider a $GK_{(B)}$- QR- affinity connected space. In view of (3.5), the condition (2.1) becomes

(3.6) $K^i_{jkh(l)(m)(n)(s)} = a_{lmns} K^i_{jkh} + b_{lmns}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$.

Transvecting (3.6) by g_{iP}, using (1.7b) and (1.9), we get

(3.7) $K_{jpkh(l)(m)(n)(s)} = a_{lmns} K^i_{jpkh}$

$+ b_{lmns}(g_{Ph} g_{jk} - g_{Pk} g_{jh})$.

Theorem (3.1). In $GK_{(B)}$- QR- affinity connected space, the Cartan's fourth curvature tensor K^i_{jkh} and its associative curvature tensor K_{jpkh} are $G_{(B)} - QR$.

Differentiating equation (1.10c) and (1.10d) covaraintly with respect to x^l, x^m , x^n and x^s , in sense of Berwald,using (3.6), (3.7) and the symmetric property of the metric tensor, we get

(3.8) $a_{lmns}\, K^i_{jkh} + a_{lmns} K^i_{khj} + a_{lmns} K^i_{hjk} = 0$.

(3.9) $a_{lmns} K^i_{jpkh} + a_{lmns} K^i_{jpkh} + a_{lmns} K^i_{jpkh} = 0$

or

(3.10) $K^i_{jkh} + K^i_{khj} + K^i_{hjk} = 0$.

(3.11) $K^i_{jpkh} + K^i_{jpkh} + K^i_{jpkh} = 0$, where $a_{lmns} \neq 0$.

Theorem (3.2). In $GK_{(B)}$- QR-affinity connected space, the Cartan's fourth curvature tensor K^i_{jkh} and its associative curvature tensor K_{jpkh} satisfey the identity Bianchi.

Contracting the indices i and h in (3.6), using (1.7c) and (1.11a) we get

(3.12) $K_{jk(l)(m)(n)(s)} = a_{lmns} K_{jk} + (n-1) b_{lmns}\, g_{jk}$.

Transvecting (3.12) by g^{jk}, using (1.11c) and (1.7c), we get

(3.13) $K_{(l)(m)(n)(s)} = a_{lmns} K + n(n-1)\, b_{lmns}$.

The equations (3.12) and (3.13) show that the Ricci tensor K_{jk} and the scalar curvature K cannot vanish, because the vanishing of any one of them would imply $b_{lmns} = 0$; a contradiction.

Theorem (3.3). In $GK_{(B)}$- QR-affinity connected space, the Ricci tensor K_{jk} and its scalar curvature tensor K are non-vanishing.

Transvecting (3.6) by $\dot{x}^j$, using (1.6) and (1.12b), we get

(3.14) $H^i_{kh(i)(m)(n)(s)} = a_{lmns} H^i_{kh} + b_{lmns}(\delta^i_h \dot{x}_k - \delta^i_k \dot{x}_h)$.

Contracting the indices i and h in equation (3.14), using (1.16b), we get

(3.15) $H_{k(l)(m)(n)(s)} = a_{cmns} H_{kh} + (n-1) b_{lmns} \dot{x}_k$

Transvecting (3.15) by $\dot{x}^k$, using (1.16c), (1.7i) and (1.7c), we get

(3.16) $H_{(l)(m)(n)(s)} = a_{lmns} H + b_{lmns}\, F^2$.

The equations (3.15) and (3.16) show that curvature vector H_K and curvature scalar cannot vanish, because the vanishing of any one of them would imply $b_{lmns} = 0$.

Theorem (3.4). In $GK_{(B)}$- QR-affinity connected space, the curvature H_k and the curvature scalar H are non-vanishing.

Transvecting (3.14) by g^{kh}, using (1.7d) and (1.7i), we get

(3.17) $(H^i_{kh} g^{kh})_{(l)(m)(n)(s)} = a_{lmns} H^i_{kh}\, g^{kh}$.

Theorem (3.5). In $GK_{(B)}$- QR-affinity connected space, the tensor $H^i_{kh} g^{kh}$ behaves as quad-recurrence.

Differentiating (1.17a) covriantly with respect to x^m, x^n and x^s, we get

$$\left(H^i_{kh}\right)_{(l)(m)(n)(s)} + \left(H^i_{hl}\right)_{(l)(m)(n)(s)} + (H^i_{lk})_{(h)(m)(n)(s)} = 0 \ .$$

Transvecting above equation by g^{kh}, we get

$$\left(H^i_{kh} g^{kh}\right)_{(l)(m)(n)(s)} + \left(H^i_{hl} g^{kh}\right)_{(l)(m)(n)(s)} + (H^i_{lk} g^{kh})_{(h)m)(n)(s)} = 0 \ .$$

Using (3.17) in above equation, we get

(3.18) $a_{lmns}\, H^i_{kh} g^{kh} + a_{kmns} H^i_{hl} g^{kh} + a_{hmns} H^i_{lk} g^{kh} = 0$.

Contracting the indices i and l in equation (3.18) using (1.7d), we get

$$a_{imns} H^i_{kh} g^{kh} + a_{kmns} H_h g^{kh} - a_{hmns} H_k g^{kh} = 0$$

or

$$a_{imns} H^i_{kh} g^{kh} = a_{hmns} H_k g^{kh} - a_{kmns} H_h g^{kh}.$$

Above equation can be written as

(3.19) $H^i_{kh} g^{kh} = (a^i_h H_k g^{kh} - a^i_k H_h g^{kh})$,
where $a^i_h = \frac{a_{hmns}}{a_{imns}}$ and $a^i_h = \frac{a_{kmns}}{a_{imns}}$.

Theorem (3.6). In $GK_{(B)}$- QR-affinity connected space, the tensor $H^i_{kh} g^{kh}$ satisfies the identity (3.18) and defined by formula (3.19) .

By cyclic rotation of the indices k , h , l and m in (3.18), we get

(3.20) $a_{mkns} H^i_{hl} g^{kh} + a_{hkns} H^i_{lm} g^{kh} + a_{lkns} H^i_{mh} g^{kh} = 0$.

(3.21) $a_{khns} H^i_{lm} g^{kh} + a_{lhns} H^i_{mk} g^{kh} + a_{mhns} H^i_{kl} g^{kh} = 0$.

(3.22) $a_{hlns} H^i_{mk} g^{kh} + a_{mlns} H^i_{kh} g^{kh} + a_{klns} H^i_{hm} g^{kh} = 0$.

By using the skew-symmetric property of the h(w) – tersion tensor H^i_{kh} in its lower indices and adding (3.18), (3.20), (3.21) and (3.22), we get

(3.23) $(a_{lmns} + a_{mlns})\, H^i_{kh} g^{kh} + (a_{mkns} + a_{kmns})\, H^i_{hl} g^{kh} + (a_{khns} + a_{hkns})\, H^i_{lm} g^{kh} + (a_{hlns} + a_{lhns})\, H^i_{mk} g^{kh} + (a_{hmns} - a_{mhns})\, H^i_{lk} g^{kh} + (a_{lkhs} - a_{klns}) H^i_{mh} g^{kh} = 0$.

If the recurrence covariant tensor a_{lmns} is symmetric in the first and second indicator, then the above equation can be written as

(3.24) $a_{lmns} H^i_{kh} g^{kh} + a_{mkns} H^i_{hl} g^{kh} + a_{khns} H^i_{lm} g^{kh} + a_{hlns} H^i_{mk} g^{kh} = 0$.

Theorem (3.7). In $GK_{(B)}$- QR-affinity connected space, if the recurrence covariant tensor a_{lmns} is symmetric, then the tensor $H^i_{kh} g^{kh}$ satisfies the identity(3.24).

And if the recurrence covariant tensor a_{lmns} is skew-symmetric in the first and second indicator, then equation (3.23), becomes

(3.25) $a_{hmns} H^i_{lk} g^{kh} + a_{lkns} H^i_{mh} g^{kh} = 0$

Corollary (4.2.1). In $GK_{(B)}$- QR-affinity connected space, if the recurrence covariant tensor a_{lmns} is skew-symmetric, then the tensor $H^i_{kh}g^{kh}$ satisfies the identity(3.25).

By cyclic rotation of the indices l, m, n, k and h in (3.24), we get

$$(3.26) \quad a_{mnks}H_{hl}g^{kh} + a_{nhks}H^i_{lm}g^{kh} + a_{hlks}H^i_{mn}g^{kh} + a_{lmks}H^i_{nh}g^{kh} = 0 .$$

$$(3.27) \quad a_{nkhs}H_{lm}g^{kh} + a_{klhs}H^i_{mn}g^{kh} + a_{lmhs}H^i_{kh}g^{kh} + a_{mnhs}H^i_{kl}g^{kh} = 0 .$$

$$(3.28) \quad a_{khls}H_{mn}g^{kh} + a_{hmls}H^i_{hk}g^{kh} + a_{mnls}H^i_{kh}g^{kh} + a_{hkls}H^i_{hm}g^{kh} = 0 .$$

$$(3.29) \quad a_{hlms}H_{hk}g^{kh} + a_{lhms}H^i_{kh}g^{kh} + a_{nkms}H^i_{hl}g^{kh} + a_{khms}H^i_{lh}g^{kh} = 0 .$$

And adding (3.24), (3.26), (3.27), (3.28) and (3.29), we get

$$\begin{aligned}(3.30) \quad & [(a_{lmns} + a_{lnms})\, H^i_{kh}g^{kh} \\ & + (a_{mkns} + a_{mnks})\, H^i_{hl}g^{kh} \\ & + (a_{nhks} + a_{hkns})\, H^i_{lm}g^{kh} \\ & + (a_{klhs} + a_{khls})\, H^i_{mn}g^{kh} \\ & + (a_{hmls} + a_{hlms})\, H^i_{nk}g^{kh}] \\ & + [a_{nkms}H^i_{hl}g^{kh} + a_{khms}H^i_{ln}g^{kh} \\ & + a_{lmhs}H^i_{hk}g^{kh} + a_{nmls}H^i_{kh}g^{kh}] \\ & + [a_{khns}H^i_{lm}g^{kh} + a_{hlns}H^i_{mk}g^{kh} \\ & + a_{mnhs}H^i_{kl}g^{kh}] \\ & + [a_{hlks}H^i_{mn}g^{kh} + a_{lmks}H^i_{nh}g^{kh} \\ & + a_{nkls}H^i_{hm}g^{kh}] = 0\end{aligned}$$

If the recurrence covariant tensor a_{lmns} is symmetric in first, second and third indicators, then the above equation can be written as

$$\begin{aligned}(3.31) \quad 2[& a_{mnks}H^i_{hl}g^{kh} + a_{nkhs}H^i_{lm}g^{kh} + a_{khls}H^i_{mn}g^{kh} \\ & + a_{hlms}H^i_{nk}g^{kh} + a_{lmns}H^i_{kh}g^{kh}] \\ & + [a_{nkms}H^i_{hl}g^{kh} + a_{khms}H^i_{ln}g^{kh} \\ & + a_{hlms}H^i_{hk}g^{kh} + a_{lnms}H^i_{kh}g^{kh}] \\ & + [a_{khns}H^i_{lm}g^{kh} + a_{lhns}H^i_{nk}g^{kh} \\ & + a_{mhns}H^i_{hl}g^{kh}] \\ & + [a_{hlks}H^i_{mn}g^{kh} + a_{mlks}H^i_{nh}g^{kh} \\ & + a_{nkls}H^i_{hm}g^{kh}] = 0\end{aligned}$$

By using equations (3.18) and (3.24) in the above equation, we get

$$(3.32) \quad a_{lmns}\, H^i_{kh}\, g^{kh} + a_{mnks}\, H^i_{hl}\, g^{kh} + a_{nkhs}\, H^i_{lm}\, g^{kh} + a_{khls}\, H^i_{mn}\, g^{kh} + a_{hlms}\, H^i_{hk}\, g^{kh} = 0 .$$

Theorem (3.8). In $GK_{(B)}$- QR-affinity connected space, if the recurrence covariant tensor a_{lmns} is symmetric in first, second and third indicators, then the tensor $H^i_{kh}g^{kh}$ satisfies the identity(3.32).

4. Conclusion

In this paper, we have studied the generalized recurrent Finsler spaces of fourth order with a non-metric Finsler connection. We have obtained the expressions of the fourth order recurrent curvature tensor and the h- fourth order recurrent curvature tensor. We have also studied the properties of these tensors and obtained some interesting results.

Future work: In the future, we plan to study the generalized recurrent Finsler spaces of higher order with a non-metric Finsler connection. We also plan to study the applications of these spaces to other areas of mathematics and physics.

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مقالة بحثية

حول تعميم المؤثر $K_{(B)}$ – رباعي الاشتقاق في فضاء فينسلر

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المُلخّص

في هذا البحث، نقوم بدراسة فضاءات فينسلر رباعي الاشتقاق المعممة من الرتبة الرابعة مع اتصال فينسلر غير متري. ونحصل على صيغ جبرية عن موتر الانحناء المتكرر K من الرتبة الرابعة و موتر الانحناء المتكرر h من الرتبة الرابعة .كما نقوم بدراسة خواص هذه المتجهات ونحصل على بعض النتائج المهمة.

الكلمات المفتاحية: فضاء فنسلر المتكرر، فضاء فينسلر المتكرر من الرتبة الرابعة، اتصال فينسلر غير متري، موتر الانحناء.

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