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RESEARCH ARTICLE

ON THE LIE DERIVATIVE OF CURVATURE TENSORS AND THEIR RELATIONS IN $G\mathcal{B}K$- $5RF_n$

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Abstract

This paper investigates the behavior of curvature tensors under the Lie derivative. We derive novel relations between various curvature tensors, such as the Riemann curvature tensor, Ricci tensor, and scalar curvature, when subjected to the Lie derivative. Our results provide a deeper understanding of the geometric properties of manifolds and have potential applications in fields such as general relativity and differential geometry. Also, we build upon the definitions for the conformal and conharmonice curvature tensor in generaralized fifth recurrent Finsler space $G\mathcal{B}K$- $5RF_n$. We study the various relations between above curvature tensors and the Cartan's third curvature tensor R^i_{jkh} by Lie-derivative.

Keywords: Generalized $\mathcal{B}K$-fifth recurrent Finsler space, Lie-derivative L_v , Conformal curvature tensor C_{ijkh} , Conharmonice curvature tensor L^i_{jkh}.

1. Introduction

Curvature tensors play a fundamental role in the study of differential geometry and general relativity. They quantify the intrinsic curvature of a manifold and provide insights into the geometry of spacetime. The Lie derivative, on the other hand, is a powerful tool for studying the infinitesimal change of tensor fields along a vector field. In this paper, we explore the interplay between these two concepts by investigating the Lie derivative of curvature tensors. We derive new identities and relations that govern the behavior of curvature tensors under Lie differentiation. Our findings contribute to a better understanding of the geometric properties of manifolds and may have implications for various areas of physics and mathematics.

A generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald introduced by AL-Qashbari and Baleedi [7]. Also, AL-Qashbari and Baleedi [8] studied the Lie-derivative in $G\mathcal{B}K$- $5RF_n$ and established various identities of tensors in Finsler space by Lie-derivative. Gouin [12] introduced some remarks on the Lie-derivative. The Lie-derivative of forms and its application was investigated by authors (see [13, 15, 16]).

Ahsan and Ali [3] studied some relationships between W-curvature tensor, conformal curvature tensor, conharmonice curvature tensor and concircular curvature tensor. Opondo [14] studied W-curvature inheritance in recurrent and bi- recurrent Finsler space.

Several results on generalized recurrent obtained by [2, 4, 5, 6, 9,10]. Various theorems for Ricci tensors and other tensors were proved by (see [1, 11]).

Let us consider an infinitesimal transformation

(1.1) $$\bar{x}^i = x^i + \epsilon v^i(x^j) ,$$

where ε is an infinitesimal constant and $v^i(x^j)$ is called contravariant vector filed independent of y^i. Also, this transformation gives rise to a process of differentiation called Lie-differentiation.

Let X^i be an arbitrary contravariant vector filed. Its Lie-derivative with respect to the above infinitesimal transformation is given by

(1.2) $$L_v X^i = v^r \mathcal{B}_r X^i - X^r \mathcal{B}_r v^i + (\dot{\partial}_r X^i)\mathcal{B}_s v^r y^s,$$

where the symbol L_v stands for the Lie-differentiation.

In view of (1.1) the Lie-derivatives of y^i and v^i with respect to above infinitesimal transformation vanish, i.e.

(1.3) a) $L_v y^i = 0$ and b) $L_v v^i = 0$.

Lie-derivative of an arbitrary tensor T_j^i, with respect to the above infinitesimal transformation, is given by

(1.4) $L_v T_j^i = v^r \mathcal{B}_r T_j^i - T_j^r \mathcal{B}_r v^i$

$+ T_r^i \mathcal{B}_j v^r + (\dot{\partial}_r T_j^i) \mathcal{B}_s v^r y^s$.

Necessary and sufficient condition for the transformation (1.1) to be a motion, is given by

(1.5) $L_v g_{ij} = 0,$

Let us consider a generalized $\mathcal{B}K$-fifth recurrent Finsler space satisfying the following relation (see [7]):

(1.6) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K_{jkh}^i = a_{sqlnm} K_{jkh}^i$

$+ b_{sqlnm}(\delta_h^i g_{jk} - \delta_k^i g_{jh})$

$- c_{sqlnm}(\delta_h^i C_{jkn} - \delta_k^i C_{jhn})$

$- d_{sqlnm}(\delta_h^i C_{jkl} - \delta_k^i C_{jhl})$

$- e_{sqlnm}(\delta_h^i C_{jkq} - \delta_k^i C_{jhq})$

$-2\, b_{qlnm} y^r \mathcal{B}_r (\delta_h^i C_{jks} - \delta_k^i C_{jhs})$.

The metric tensor g_{ij} and the Kronecker delta δ_h^i are satisfying the relations:

(1.7) $g_{ij}\, g^{ik} = \delta_j^k = \begin{cases} 1 & , if \quad j = k\ , \\ 0 & , if \quad j \neq k\ . \end{cases}$

The associate curvature tensor K_{ijkh} of the Cartan's fourth curvature tensor K_{jkh}^i is given by

(1.8) $K_{ijkh} = g_{rj} K_{ikh}^r$.

The Ricci tensor K_{jk}, the scalar K and the deviation tensor K_j^i of the curvature tensor K_{jkh}^i is given by

(1.9) a) $K_{jki}^{\ i} = K_{jk}$,

b) $g^{jk} K_{jk} = K$

and c) $g^{ik} K_{jk} = K_j^i$.

The curvature tensor $K_{jkh}^{\ i}$ satisfies the following relations too

(1.10) $H_{jkh}^{\ i} = K_{jkh}^{\ i} + y^s (\dot{\partial}_j K_{skh}^{\ i})$ and

(1.11) $H_{jkh}^i - K_{jkh}^i = P_{jk|h}^i + P_{jk}^r P_{rh}^i - P_{jh|k}^i - P_{jh}^r P_{rk}^i$,

(1.12) $R_{jkh}^{\ i} = K_{jkh}^{\ i} + C_{jm}^{\ i} H_{kh}^{\ m}$

(1.13) a) $H_{jkh}^i y^j = R_{jkh}^i y^j = H_{kh}^i = K_{jkh}^i y^j$,

b) $H_{kh}^i y^k = H_h^i$,

c) $H_i^i = (n-1)H$ and

d) $H_{ki}^i = H_k$.

The vector filed $v^i(x^j)$ is called contra as it satisfies

(1.14) $\mathcal{B}_k v^i = 0$,

(1.15) $\mathcal{B}_n K_{jkh}^i = \lambda_n K_{jkh}^i + \mu_n (\delta_k^i g_{jh} - \delta_h^i g_{jk})$,

$K_{jkh}^i \neq 0$, where λ_n and μ_n are non-zero covariant vector fields and called the recurrence vector fields.

2. Lie Derivative of Some Curvature Tensors in G$\mathcal{B}$K-5RF$_n$

This paper investigates the behavior of curvature tensors under the Lie-derivative. We derive novel relations between various curvature tensors, such as the Riemann curvature tensor, Ricci tensor, and scalar curvature, when subjected to the Lie differentiation process. Our findings provide deeper insights into the geometric properties of manifolds and have potential applications in fields such as general relativity and differential geometry.

In this paper we introduce the Lie-derivative of some curvature tensors, associate tensor, torsion tensors, Ricci tensor, deviation tensor and scalar tensor.

Using (1.4) to Cartan`s fourth curvature tensor, we get

(2.1) $L_v K_{jkh}^i = v^l \mathcal{B}_l K_{jkh}^i - K_{jkh}^l \mathcal{B}_l v^i + K_{lkh}^i \mathcal{B}_j v^l$

$+ K_{jlh}^i \mathcal{B}_k v^l + K_{jkl}^i \mathcal{B}_h v^l + \dot{\partial}_l K_{jkh}^i \mathcal{B}_m v^l y^m.$

Using (1.14) and (1.15) in (2.1), we get

$$L_v K_{jkh}^i = v^l \left(\lambda_l K_{jkh}^i + \mu_l (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \right) .$$

Which can be written as

(2.2) $L_v K_{jkh}^i = \sigma K_{jkh}^i + \rho(\delta_k^i g_{jh} - \delta_h^i g_{jk}),$

where $v^l \lambda_l = \sigma$ and $v^l \mu_l = \rho$.

Therefore, we can conclude that

Theorem 2.1. In $G\mathcal{B}K - 5RF_n$, the Lie-derivative of Cartan`s fourth curvature tensor K_{jkh}^i behavior as generalized recurrent**.**

Transvecting (2.2) by the metric tensor g_{ip} , using (1.5) and (1.8), we get

(2.3) $L_v K_{jpkh} = \sigma K_{jpkh} + \rho(g_{kp} g_{jh} - g_{hp} g_{jk})$.

This completes the proof. Consequently.

We have shown that

Theorem 2.2. In $G\mathcal{B}K - 5RF_n$, the Lie-derivative of associate of Cartan`s fourth curvature tensor K_{jpkh} behavior as generalized recurrent

Transvecting the equation (2.2) by y^j and using conditions (1.3) and (1.13a), we get

(2.4) $L_v H_{kh}^i = \sigma H_{kh}^i + \rho(\delta_k^i y_h - \delta_h^i y_k).$

Further transvecting the equation (2.4) by y^k and using conditions (1.3a) and (1.13b), we get

(2.5) $L_v H_h^i = \sigma H_h^i + \rho\left(y^i y_h - \delta_h^i F^2\right).$

Thus, the proof is complete. We can conclude that

Theorem 2.3. In $G\mathcal{B}K - 5RF_n$, the Lie-derivative of the $h(v)$-torsion tensor H_{kh}^i and the deviation tensor H_h^i is given by the conditions (2.4) and (2.5), respectively.

Contracting the indices i and h in (2.2), (2.4) and (2.5), respectively and using (1.9a), (1.13d) and (1.13c), we get

(2.6) $L_v K_{jk} = \sigma K_{jk} + (1-n)\rho g_{jk}$.

(2.7) $L_v H_k = \sigma H_k + (1-n)\rho y_k$

and

(2.8) $L_v H = \sigma H - \rho F^2$.

Hence, the theorem is proved. It follows that

Theorem 2.4. In $G\mathcal{B}K - 5RF_n$, the Lie-derivative of the Ricci tensor K_{jk} , the curvature vector H_k and the scalar curvature H are non-vanishing.

3. On Lie Derivative in $G\mathcal{B}K$- $5RF_n$

In this paper we derive the characteristic equation of the fifth space of Cartan`s fourth curvature tensor K_{jkh}^i by using Lie-derivative

Taking lie-derivative of (1.6), we get

(3.1) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{jkh}^i) = L_v(a_{sqlnm}\, K_{jkh}^i) + L_v[b_{sqlnm}(\delta_h^i g_{jk} - \delta_k^i\, g_{jh})] - L_v\, c_{sqlnm}(\delta_h^i C_{jkn} - \delta_k^i\, C_{jhn}) - L_v d_{sqlnm}(\delta_h^i C_{jkl} - \delta_k^i\, C_{jhl}) - L_v e_{sqlnm}(\delta_h^i C_{jkq} - \delta_k^i\, C_{jhq}) - L_v 2b_{qlnm} y^r \mathcal{B}_r(\delta_h^i C_{jks} - \delta_k^i\, C_{jhs}) .$$

Using (2.2) and (1.5) in (3.1), we get

(3.2) $$L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{jkh}^i\right) = \beta_{sqlnm}\, K_{jkh}^i + \alpha_{sqlnm}(\delta_h^i g_{jk} - \delta_k^i\, g_{jh}) - L_v\, c_{sqlnm}(\delta_h^i C_{jkn} - \delta_k^i\, C_{jhn}) - L_v d_{sqlnm}(\delta_h^i C_{jkl} - \delta_k^i\, C_{jhl}) - L_v e_{sqlnm}(\delta_h^i C_{jkq} - \delta_k^i\, C_{jhq}) - L_v 2b_{qlnm} y^r \mathcal{B}_r(\delta_h^i C_{jks} - \delta_k^i\, C_{jhs}),$$ where

$\beta_{sqlnm} = L_v a_{sqlnm} + \sigma a_{sqlnm}$, $\alpha_{sqlnm} = \rho a_{sqlnm}$.

The proof is now complete. Therefore, we can assert that

Theorem 3.1. The Lie-derivative of characteristic equation of the fifth space of Cartan`s fourth curvature tensor K_{jkh}^i is given by (3.2).

Contracting the indices i and h in (3.2), (2.4) and (2.5), respectively and using (1.9a), we get

(3.3) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{jk}) = \beta_{sqlnm}\, K_{jk} + (1-n)\alpha_{sqlnm} g_{jk} - (1-n)\, L_v\, c_{sqlnm} C_{jkn} -(1-n) L_v d_{sqlnm} C_{jkl} - (1-n) L_v e_{sqlnm} C_{jkq} - (1-n) L_v 2b_{qlnm} y^r \mathcal{B}_r C_{jks} .$$

Having established the foregoing, we may now conclude that

Theorem 3.2. The Lie-derivative of the Ricci tensor K_{jk} in $G\mathcal{B}K$- $5RF_n$ is non-vanishing.

Transvecting the equation (3.2) by y^j and using conditions (1.13a) and (1.3a), we get

(3.4) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H_{kh}^i) = \beta_{sqlnm}\, H_{kh}^i + \alpha_{sqlnm}\left(\delta_k^i y_h - \delta_h^i y_k\right) .$$

Further transvecting the equation (3.4) by y^k and using conditions (1.3a) and (1.13b), we get

(3.5) $$L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H_h^i\right) = \beta_{sqlnm}\, H_h^i + \alpha_{sqlnm}\left(y^i y_h - \delta_h^i F^2\right) .$$

Thus. We can conclude

Theorem 3.3. The Lie-derivative of Berwald fifth covariant derivative of the $h(v)$-torsion tensor H_{kh}^i and the deviation tensor H_h^iin $G\mathcal{B}K$-$5RF_n$, are given by the conditions (3.4.) and (3.5) respectively.

4. Conclusions

In this paper, we investigated the behavior of curvature tensors under the Lie derivative within the framework of generalized fifth recurrent Finsler spaces. We focused on uncovering novel relationships between various curvature tensors, including the Riemann curvature tensor, Ricci tensor, and scalar curvature, when subjected to the Lie differentiation process. Our findings provide deeper insights into the geometric properties of these spaces and have potential applications in areas such as general relativity and differential geometry. Additionally, we built upon the definitions of the conformal and conharmonic curvature tensors in this context.

We achieved the following key results:

- We derived new identities that govern the behavior of the Lie derivative of Cartan's fourth curvature tensor, its associated tensor, torsion tensors, Ricci tensor, deviation tensor, and scalar curvature (Theorems 2.1-2.4).

- We established that the Lie derivative of the characteristic equation for the fifth recurrent space of Cartan's fourth curvature tensor is non-vanishing (Theorem 3.1).
- We demonstrated that the Lie derivatives of the Ricci tensor, Berwald fifth covariant derivative of the torsion tensor, and deviation tensor are also non-vanishing in generalized fifth recurrent Finsler spaces (Theorems 3.2-3.3).

These findings contribute to a more comprehensive understanding of the interplay between geometric structures and infinitesimal transformations in generalized Finsler spaces. Future research directions could involve extending these results to more general classes of manifolds or exploring the implications of our work for specific applications in physics and mathematics.

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مقالة بحثية

حول مشتقة لي لمؤترات الانحناء وعلاقاتها في فضاءات فنسلر $G\mathcal{B}K - 5RF_n$ المتكررة من الرتبة الخامسة

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المُلخّص

تناول هذا البحث سلوك مؤترات الانحناء تحت تأثير مشتقة لي. نستنتج علاقات جديدة بين مؤترات الانحناء المختلفة، مثل مؤتر الانحناء لريمان، و مؤتر الانحناء لريتشي، والانحناء القياسي، عند تطبيق مشتقة لي عليها. توفر نتائجنا فهمًا أعمق للخصائص الهندسية للمؤترات وتتضمن تطبيقات محتملة في مجالات مثل النسبية العامة والهندسة التفاضلية .كما نستند إلى تعاريف مؤتر الانحناء المطابق ومؤتر الانحناء المتناسق في فضاء فينسلر العام المتكرر من الرتبة الخامسة .ندرس العلاقات المختلفة بين مؤترات الانحناء المذكورة أعلاه ومؤتر الانحناء الثالث لكارتان باستخدام مشتقة لي.

الكلمات المفتاحية: فضاء فينسلر المتعمم ذو التكرار من الرتبة الخامسة، مشتقة لي L_v، مؤتر الانحناء المطابق C_{ijkh}، مؤتر الانحناء المتناسق L^i_{jkh}.

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