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RESEARCH ARTICLE

GENERALIZATION OF WELY'S PROJECTIVE CURVATURE TENSOR IN FINSLER SPACES: A STUDY ON GENERALIZED W-RECURRENT, BIRECURRENT, AND RICCI TENSORS

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Abstract

In this study, we explore the generalization of Wely's projective curvature tensor W^i_{jkh} satisfying a specific recurrence relation (3.1), which defines a generalized W-recurrent space, denoted as $G^{2nd}\ W_{|h} - BRF_n$. By investigating the conditions under which this tensor satisfies the generalized recurrence relations, we define and analyze the properties of generalized W-recurrent, birecurrent, and Ricci tensors in the context of second-order covariant derivatives. We derive a series of equations that describe the behavior of these tensors, including the covariant derivative expressions and their relationships with scalar curvatures and deviation tensors. A key contribution is the proof of various theorems that relate the generalized recurrence conditions to torsion tensors and curvature properties. Specifically, we show that the Ricci tensor and associated tensors exhibit generalized birecurrent Finsler space behavior under certain conditions. The results further provide insights into the torsion and deviation tensors, highlighting their role in the overall curvature structure of Finsler spaces. These findings extend the theory of recurrent spaces, offering a comprehensive framework for understanding higher-order curvature phenomena in Finsler geometry.

Keywords: Generalized Recurrent and Birecurrent Finsler Space, Covariant derivative, Weyl Tensor W^i_{jkh} , Cartan's Curvature, Torsion Tensor and Ricci Tensor.

I. Introduction

Finsler geometry, as an extension of Riemannian geometry, has become a vital tool in both pure and applied mathematics, with applications spanning physics, engineering, and other fields. The study of curvature tensors in Finsler spaces has been a focal point for researchers seeking to understand the geometrical structures that characterize these spaces. The foundational works of many scholars have shaped our understanding of curvature properties, particularly with respect to various generalized and recurrent structures in Finsler spaces.

A key advancement in this area was made [1], who explored the w*-curvature tensor in the context of relativistic space-times, providing new insights into the behavior of curvature in spacetime. [2] furthered this study by introducing the curvature tensor for spacetime in general relativity, offering a more detailed look at the interaction between geometry and physics. These studies have significantly contributed to our understanding of curvature in spaces with non-constant curvature. In the realm of Finsler spaces, Al-Qashbari and collaborators have made substantial contributions to the study of recurrent structures and higher-order generalizations. [3] presented a study on recurrent Finsler structures defined by special curvature tensors, shedding light on their higher-order generalizations. Similarly, [4] examined generalized BK-recurrent Finsler spaces, extending the scope of previous research on these recurrent properties and highlighting their importance in understanding complex geometric structures. Additionally, the exploration of generalized birecurrent Finsler spaces by [5], [6] has provided a more profound comprehension of the geometric properties associated with mixed covariant derivatives in Cartan's sense. Further research on curvature tensors in specific Finsler spaces, such as the studies of [7], [8], [9], [10] on second-order generalized curvature tensors, and [13] on higher-order recurrent Finsler spaces with Berwald's curvature tensor, has expanded the framework for understanding the complex interactions between curvature and geometric structures in higher-dimensional Finsler spaces. The systematic review by [14] on various special Finsler spaces has also contributed to this comprehensive body of knowledge, offering a valuable resource for researchers interested in the diversity of curvature-related properties in Finsler geometry.

Previous research in the field of Finsler geometry has significantly contributed to the understanding of generalized recurrent spaces, curvature properties, and their applications. [12] explored certain generalized BK-recurrent Finsler spaces, contributing to the deeper understanding of Finsler geometry through the study of their curvature tensors. In parallel, [15] focused on generalized birecurrent spaces, further enhancing the mathematical foundation of these structures. Pandey, Saxena, and [16] also contributed to the study of generalized H-recurrent spaces, exploring their geometric characteristics. [17], in his seminal work, provided a comprehensive overview of the differential geometry of Finsler spaces, laying the groundwork for subsequent studies in the field. Collectively, these studies have paved the way for further exploration of Finsler spaces, particularly in the context of recurrent spaces and curvature tensors.

This paper aims to extend these studies by focusing on the generalized curvature tensors in Finsler spaces, particularly in relation to the recurrence properties and higher-order generalizations of these structures. By building upon the work of these key researchers, we seek to provide new insights into the nature of curvature and its implications for the classification and analysis of Finsler spaces.

In the field of differential geometry, Finsler spaces represent an essential extension of Riemannian geometry, with applications in both physics and pure mathematics. This paper introduces and explores the generalized W-recurrent space, a specific class of Finsler space, where the curvature tensor satisfies a novel condition, as defined by equation (3.1). The study of such generalized spaces is crucial for understanding the behavior of curvature in higher-dimensional geometries, particularly in spaces with non-constant curvature.

This research derives several fundamental relationships, including the transvection of curvature tensors into higher-dimensional spaces, the covariant derivatives of torsion and deviation tensors, and the resulting theorems that govern the behavior of these tensors in Finsler geometry. We specifically examine the generalized birecurrent property of curvature tensors, extending the scope of Finsler space studies and contributing new insights into the classification of such spaces.

The key result of this work is the derivation of explicit conditions for the torsion and Ricci tensors, as well as the curvature relations (3.7) and (4.19), which provide deep connections between the curvature of the space and the underlying geometric structure. Furthermore, this paper demonstrates that under specific conditions, Finsler spaces with generalized birecurrent properties exhibit unique geometric characteristics, offering a new direction for future research in curvature and torsion analysis in Finsler geometry.

Two vectors y_i and y^i meet the following conditions

a) $y_i = g_{ij}\, y^j$,

b) $y_i\, y^i = F^2$,

c) $\delta^k_j\, y^j = y^k$,

d) $g_{ir}\, \delta^i_j = g_{rj}$,

e) $g^{jk}\, \delta^i_k = g^{ji}$,

f) $\dot{\partial}_i\, y^i = 1$ and

g) $\dot{\partial}_j\, y_h = g_{jh}$. (1.1)

The quantities g_{ij} and g^{ij} are related by

a) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1\ , & if \quad i = k\ , \\ 0\ , & if \quad i \neq k\ . \end{cases}$

b) ${g^{jk}}_{|h} = 0$ and

c) $g_{ij|h} = 0$. (1.2)

Tensor C_{ijk} is known as (h)hv-torsion tensor defined by

a) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2$ and

b) $C_{ijk}\, y^i = C_{ijk}\, y^j = C_{ijk}\, y^k = 0$. (1.3)

The vector y^i and metric function F are vanished identically for Cartan's covariant derivative.

a) $F_{|h} = 0$ and

b) ${y^i}_{|h} = 0$. (1.4)

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j, successively, we get

$$X^i_{|k|j} = \partial_j\left(X^i_{|k}\right) - \left(X^i_{|r}\right)\Gamma^{*r}_{kj} + \left(X^r_{|k}\right)\Gamma^{*i}_{rj} - \left(\partial_j X^i_{|k}\right)\Gamma^{*i}_{js}\, y^s\ . \quad (1.5)$$

In view (1.5) and by taking skew-symmetric part with respect to the indices j and k, we get the commutation formula for Cartan is covariant differentiation as follows:

$$X^i_{|k|j} - X^i_{|j|k} = X^r K^i_{rkj} - (\partial_r X^i)\, K^i_{skj}\, y^s\ . \quad (1.6)$$

Where, $K^i_{jkh} = \partial_j \Gamma^{*i}_{kr} + (\dot\partial_l \Gamma^{*i}_{rj})G^l_k + \Gamma^{*i}_{mj}\Gamma^{*m}_{kr}$

$-\partial_k \Gamma^{*i}_{jr} - (\dot\partial_l \Gamma^{*i}_{rk})G^l_j - \Gamma^{*i}_{mk}\Gamma^{*m}_{jr}$. (1.7)

The tensor K^i_{jkh} as defined above is called Cartan's fourth curvature tensor, this tensor is positively homogeneous of degree zero in the directional arguments y^i .

Tensor W^i_{jkh} , torsion tensor W^i_{jk} and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)} \dot\partial_j H_{[kh]}$$

$$+ \frac{\delta^i_k}{(n^2-1)}\left(n\, H_{jh} + H_{hj} + y^r \dot\partial_j H_{hr}\right)$$

$$- \frac{\delta^i_h}{(n^2-1)}\left(n\, H_{jk} + H_{kj} + y^r \dot\partial_j H_{kr}\right) , \quad (1.8)$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]}$$

$$+2\left\{\frac{\delta^i_{[j}}{(n^2-1)}\left(n\, H_{k]} - y^r H_{k]\,r}\right)\right\}. \quad (1.9)$$

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)}\left(\dot\partial_r H^r_j - \dot\partial_j H\right) y^i, \quad (1.10)$$

respectively.

The tensors W^i_{jkh} , and W^i_{jk} satisfy the following identities

a) $W^i_{jkh}\, y^j = W^i_{kh}$,

b) $W^i_{kh}\, y^k = W^i_h$,

c) $W^i_{jki} = W_{jk}$ and

d) $g_{ir}\, W^i_{jkh} = W_{rjkh}$. (1.11)

Also, if we suppose that the tensor W^i_j and W_{jk} satisfy the following identities

a) $W^i_k\, y^k = 0$,

b) $W^i_i = 0$,

c) $g_{ir}\, W^i_j = W_{rj}$,

d) $g^{jk} W_{jk} = W$ and

e) $W_{jk}\, y^k = 0$. (1.12)

Cartan's third curvature tensor R^i_{jkh} , Ricci tensor R_{jk} , H^i_{kh}, the vector H_k and scalar curvature H are defined as

a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + (\Gamma^{*i}_{ljk})\, G^l_h + C^i_{jm}(G^m_{kh} - G^m_{kl}\, G^l_h)$

$+\Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - \frac{k}{h}$,

b) $R^i_{jkh}\, y^j = H^i_{kh}$,

c) $R_{jk}\, y^j = H_k$,

d) $R_{jk}\, y^k = R_j$,

e) $R^i_i = R$,

f) $g_{ir}\, R^i_{jkh} = R_{rjkh}$,

g) $g^{jk} R_{jk} = R$,

h) $g^{jk} R^i_{jkh} = R^i_h$,

i) $R^i_{jki} = R_{jk}$,

t) $H_i\, y^i = H^i_i = (n-1)\, H$,

j) $H^i_{kh}\, y^k = H^i_h$ and

k) $H^i_{ki} = H_k$. (1.13)

The 4th curvature tensor K^l_{jkh} , the Ricci tensor K_{jk} , the vector K_k , and the scalar curvature K are defined as follows

a) $K^i_{jkh}\, y^j = H^i_{kh}$,

b) $K_{jk}\, y^j = H_k$,

c) $K_{jk}\, y^k = K_j$,

d) $g^{jk} K_{jk} = K$,

e) $K^i_{jki} = K_{jk}$ and

f) $g^{jk}\, K^i_{jkh} = K^i_h$. (1.14)

2. Preliminaries

We introduced the generalized by Cartan's covariant derivative for Wely's projective curvature tensor W^i_{jkh} was given by (see [7]).

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \quad (2.1)$$

A Finsler space F_n which the curvature tensor W^i_{jkh} satisfies the condition (2.1) is called a generalized $W_{|h}$-recurrent space and denoted by $G\, W_{|h} - RF_n$.

$|m$ is called h-covariant derivative of with respect to x^m.

Taking the h-covariant derivative of (2.1), with respect to x^l, and using (1.2c), we get

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m W^i_{jkh|l}$$

$$+\mu_{m|l}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right).$$

Using (2.1) in above equation, we get

$$W^i_{jkh|m|l} = \left(\lambda_{m|l} + \lambda_m \lambda_l\right) W^i_{jkh}$$

$$+\left(\mu_l + \mu_{m|l}\right)\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \quad (2.2)$$

The equation (2.2), can be written as

$$W^i_{jkh|m|l} = a_{ml}\, W^i_{jkh} + b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right). \quad (2.3)$$

where $a_{ml} = \lambda_{m|l} + \lambda_m \lambda_l$ and $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ are non-zero covariant tensors field of second order, respectively.

A Finsler space F_n which the curvature tensor W^i_{jkh} satisfies the condition (2.3) is called a generalized $W_{|h}$-birecurrent space and denoted by $GW_{|h} - BRF_n$.

From (1.3b), we can write (2.1) by the follows form

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \gamma_m(W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i) . \qquad (2.4)$$

3. The Extension of Generalized $W_{|h}$– Birecurrent Finsler Space

In this section, we introduce a new class of Finsler spaces, namely, generalized $W_{|h}$-birecurrent spaces. These spaces generalize the concept of birecurrence to a broader setting and exhibit interesting geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems. Our work in this section we defined $|l|m$ is covariant derivative of second order.

Using the conditions (1.3a), (1.1b), (1.1f), and (1.1g), in (2.4), we get

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(W^i_k\, g_{jh} - W^i_h g_{jk}) . \qquad (3.1)$$

A Finsler space F_n which the curvature tensor W^i_{jkh} satisfies the condition (3.1) is called the generalization generalized $W_{|h}$- recurrent space and denoted by $G^{2nd}\, W_{|h} - RF_n$.

Taking the h-covariant derivative of (3.1), with respect to x^l, we get

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m W^i_{jkh|l} + \mu_{m|l}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})_{|l} + \tfrac{1}{4}\gamma_{m|l}(W^i_k g_{jh} - W^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k\, g_{jh} - W^i_h g_{jk})_{|l}. \qquad (3.2)$$

Using (1.2c) and (3.1) in (3.2), we get

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m\left(\lambda_l W^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}\gamma_l(W^i_k g_{jh} - W^i_h g_{jk})\right) + \mu_{m|l}(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}) + \tfrac{1}{4}\gamma_{m|l}(W^i_k g_{jh} - W^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l} .$$

Or

$$W^i_{jkh|m|l} = (\lambda_{m|l} + \lambda_m\lambda_l) W^i_{jkh} + (\lambda_m \mu_l + \mu_{m|l})(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(\lambda_m\gamma_l + \gamma_{m|l})(W^i_k g_{jh} - W^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l}. \qquad (3.3)$$

The equation (3.3), can be written as

$$W^i_{jkh|m|l} = a_{ml} W^i_{jkh} + b_{ml}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}c_{ml}(W^i_k g_{jh} - W^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l}. \qquad (3.4)$$

where $a_{ml} = \lambda_{m|l} + \lambda_m\lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ and $c_{ml} = (\lambda_m\gamma_l + \gamma_{m|l})$ are non-zero covariant tensors field of second order, respectively.

Definition 3.1. In Finsler space, which the Wely's projective curvature tensor W^i_{jkh} satisfies the condition (3.4) is called the generalization generalized.

$W_{|h}$- Birecurrent specs and the tensor will be called a generalization generalized h-Birecurrent specs. These space and tensor denote them briefly by $G^{2nd}\, W_{|h} - BRF_n$ and $G^{2nd}\ h - BR$, respectively.

Result 3.1. All a $G^{2nd}\, W_{|h}$-recurrent space is a $G^{2nd}\, W_{|h}$- Birecurrent specs.

Transvecting condition to a higher dimensional space (3.4) by y^j, using (1.1a), (1.3b), (1.5b) and (1.11a), we get

$$W^i_{kh|m|l} = a_{ml} W^i_{kh} + b_{ml}(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}c_{ml}(W^i_h y_k - W^i_k y_h) + \tfrac{1}{4}\gamma_m(W^i_h\, y_k - W^i_k\, y_h)_{|l} . \qquad (3.5)$$

Again, transvecting condition to a higher dimensional space (3.5) by y^k, using (1.1b), (1.2a), (1.2c), (1.4b), (1.12a) and (1.11b), we get

$$W^i_{h|m|l} = a_{ml} W^i_h + b_{ml}(\delta^i_h F^2 - y^i y_h) + \tfrac{1}{4}c_{ml} W^i_h F^2 + \tfrac{1}{4}\gamma_m\,(W^i_h F^2)_{|l}. \qquad (3.6)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.1. In $G^{2nd}\, W_{|h} - BRF_n$, the torsion tensor W^i_{kh} and deviation tensor W^i_h are given by equations (3.5) and (3.6).

Contracting the indices space by summing over i and h in the conditions (3.4) and using (1.1d), (1.2a), (1.11c), (1.12b) and (1.12c), we get

$$W_{jk|m|l} = a_{ml}W_{jk} + b_{ml}(n-1)g_{jk}$$
$$-\tfrac{1}{4}c_{ml}W_{jk} - \tfrac{1}{4}\gamma_m W_{jk|l} \quad . \qquad (3.7)$$

Again, transvecting condition to a higher dimensional space (3.4) by g_{ir}, using (1.1d),(1.2c), (1.11d) and (1.12c) we get

$$W_{rjkh|m|l} = a_{ml}W_{rjkh} + b_{ml}\left(g_{rh}\,g_{jk} - g_{rk}\,g_{jh}\right)$$
$$+\tfrac{1}{4}c_{ml}\left(W_{rh}\,g_{jk} - W_{rk}\,g_{jh}\right)$$
$$+\tfrac{1}{4}\gamma_m\left(W_{rh}g_{jk} - W_{rk}g_{jh}\right)_{|l} . \qquad (3.8)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.2. In $G^{2nd}\,W_{|h} - BRF_n$, the Ricci W_{jk} and the associate tensor W_{rjkh} are generalized birecurrent Finsler space given by the equations (3.7) and (3.8).

Transvecting (3.7) by g^{jk}, and using (1.1e) and (1.12d), we get

$$W_{|m|l} = a_{ml}W + b_{ml}(n-1)$$
$$-\tfrac{1}{4}\,c_{ml}W - \tfrac{1}{4}\,\gamma_m W_{|l} . \qquad (3.9)$$

From conditions (3.9), we show that the curvature scalar W cannot equal to zero because if the vanishing of W would imply $a_{ml} = 0$ and $b_{ml} = 0$, that is a contradiction.

Therefore, the proof of theorem is completed, we can say

Theorem 3.3. In $G^{2nd}\,W_{|h} - BRF_n$, the scalar W in equations (3.9) is non-vanishing.

4. Exploring the Relationship Between Weyl's Curvature Tensor and Cartan's Third Curvature Tensor in Finsler Geometry

Finsler geometry, which extends the concepts of Riemannian geometry, offers a robust framework for modeling a diverse range of physical phenomena. Within Finsler spaces, the curvature properties of the space are described by various curvature tensors, with Weyl's and Cartan's third curvature tensors being of particular significance. While the individual geometric and physical implications of these tensors have been widely studied, the connection between them has not been fully explored and remains an area of active research.

This paper aims to examine the relationship between Weyl's curvature tensor and Cartan's third curvature tensor within the context of Finsler spaces. By analyzing their algebraic and geometric properties, we aim to derive new identities and inequalities that connect these two fundamental tensors. The results of this study are expected to enhance our understanding of the curvature structure of Finsler spaces and offer new insights into their potential applications, particularly in gravitational theories and cosmology.

Some properties of W^i_{jkh} curvature tensor was proposed by Ahsan and Ali [3],[4] in (2014). For $(n = 4)$ a Riemannian space, it is known that Cartan's third curvature tensor R^i_{jkh} and Wely's projective curvature tensor W^i_{jkh} are connected by the formula

$$W^i_{jkh} = R^i_{jkh} + \tfrac{1}{3}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right) . \qquad (4.1)$$

Taking the covariant derivative of (4.1), with respect to x^m and x^l in the sense of Cartan and using (1.2c), we get

$$W^i_{jkh|m|l} = R^i_{jkh|m|l} + \tfrac{1}{3}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)_{|m|l} . \qquad (4.2)$$

Using (3.4) and (4.1) in (4.2), we get

$$R^i_{jkh|m|l} = a_{ml}\left(R^i_{jkh} + \tfrac{1}{3}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)\right)$$
$$+ b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l}$$
$$+\tfrac{1}{3}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)_{|m|l}.$$

Or

$$R^i_{jkh|m|l} = a_{ml}\,R^i_{jkh} + b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$+\tfrac{1}{3}a_{ml}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)$$
$$+\tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l}$$
$$+\tfrac{1}{3}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)_{|m|l} . \qquad (4.3)$$

This shows that

$$R^i_{jkh|m|l} = a_{ml}\,R^i_{jkh} + b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} . \qquad (4.4)$$

If and only if

$$\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)_{|m|l}$$
$$= a_{ml}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right). \qquad (4.5)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.1. In $G^{2nd}\,W_{|h} - BRF_n$, Cartan's third curvature tensor R^i_{jkh} is a generalized birecurrent Finsler space if and only if the tensor $\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)$ is a generalized birecurrent Finsler space.

Transvecting condition (4.3) by y^j, using (1.1a), (1.4b), (1.13b) and (1.13c), we get

$$H^i_{kh|m|l} = a_{ml}\, H^i_{kh} + b_{ml}(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{3} a_{ml}(\delta^i_k\, H_h - y_k\, R^i_h) + \tfrac{1}{4} c_{ml}(W^i_h y_k - W^i_k y_h) + \tfrac{1}{4}\gamma_m (W^i_h y_k - W^i_k y_h)_{|l} + \tfrac{1}{3}(\delta^i_k\, H_h - y_k\, R^i_h)_{|m|l}. \quad (4.6)$$

This shows that

$$H^i_{kh|m|l} = a_{ml}\, H^i_{kh} + b_{ml}(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4} c_{ml}(W^i_h y_k - W^i_k y_h) + \tfrac{1}{4}\gamma_m (W^i_h y_k - W^i_k y_h)_{|l}. \quad (4.7)$$

If and only if

$$(\delta^i_k\, H_h - y_k\, R^i_h)_{|m|l} = a_{ml}(\delta^i_k\, H_h - y_k\, R^i_h). \quad (4.8)$$

The proof of theorem is completed, we conclude

Theorem 4.2. In $G^{2nd}\, W_{|h} - BRF_n$, the covariant derivative of the second orders for the torsion tensor H^i_{kh} is a generalized birecurrent Finsler space if and only if (4.8), holds good.

Transvecting (4.6) by y^k, using (1.1b), (1.1c), (1.4a), (1.4b), (1.12a) and (1.13j), we get

$$H^i_{h|m|l} = a_{ml} H^i_h + b_{ml}(y^i\, y_h - \delta^i_h\, F^2) + \tfrac{1}{4} c_{ml} W^i_h F^2 + \tfrac{1}{4}\gamma_m W^i_{h|l}\, F^2 + \tfrac{1}{3} a_{ml}(\, y^i H_h - F^2 R^i_h) + \tfrac{1}{3}(\, y^i H_h - F^2\, R^i_h)_{|m|l}. \quad (4.9)$$

This shows that

$$H^i_{h|m|l} = a_{ml} H^i_h + b_{ml}(\, y^i\, y_h - \delta^i_h\, F^2) + \tfrac{1}{4} c_{ml}\, W^i_h F^2 + \tfrac{1}{4}\gamma_m\, W^i_{h|l}\, F^2. \quad (4.10)$$

If and only if

$$(y^i H_h - F^2\, R^i_h)_{|m|l} = a_{ml}(y^i H_h - F^2 R^i_h). \quad (4.11)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.3. In $G^{2nd}\, W_{|h} - BRF_n$, the covariant derivative of the second orders for the deviation tensor H^i_h is a generalized birecurrent Finsler space if and only if (4.11), holds good.

Contracting the indices i and h in the equations (4.6) and (4.9), respectively and using (1.2a), (1.1a), (1.1b), (1.4a), (1.13k), (1.13t), and (1.12b), we get

$$H_{k|m|l} = a_{ml} H_k + (1-n)\, b_{ml}\, y_k + \tfrac{1}{4} c_{ml}(W^i_k y_i) + \tfrac{1}{4}\gamma_m (W^i_k y_i)_{|l} + \tfrac{1}{3} a_{ml}(H_k - y_k\, R) + \tfrac{1}{3}(H_k - y_k\, R)_{|m|l}. \quad (4.12)$$

This shows that

$$H_{k|m|l} = a_{ml} H_k + (1-n)\, b_{ml}\, y_k + \frac{1}{4} c_{ml}(W^i_k y_i) + \tfrac{1}{4}\gamma_m (W^i_k y_i)_{|l}. \quad (4.13)$$

If and only if

$$(H_k - y_k\, R)_{|m|l} = a_{ml}(\, H_k - y_k\, R). \quad (4.14)$$

And

$$H_{|m|l} = a_{ml} H + (1-n) b_{ml}\, F^2 + \tfrac{1}{3}\, (H - F^2 R)_{|m|l} + \tfrac{1}{3} a_{ml}(H - F^2 R).$$

(4.15)

This shows that

$$H_{|m|l} = a_{ml} H + (1-n) b_{ml}\, F^2. \quad (4.16)$$

If and only if

$$(H - F^2 R)_{|m|l} = a_{ml}(H - F^2 R). \quad (4.17)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.4. In $G^{2nd}\, W_{|h} - BRF_n$, vector H_k and scalar H are given in (4.13) and (3.16) if and only if the conditions (4.14) and (3.17) are holds good, respectively.

Contracting the indices i and h in the equations (4.3) and using (1.1d), (1.1b), (1.13i), (1.13e), (1.12d) and (1.12b), we get

$$R_{jk|m|l} = a_{ml} R_{jk} + (1-n)\, b_{ml}\, g_{jk} + \tfrac{1}{4} c_{ml} W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l} + \tfrac{1}{3}(\, R_{jk} - g_{jk}\, R)_{|m|l} + \tfrac{1}{3} a_{ml}(\, R_{jk} - g_{jk}\, R). \quad (4.18)$$

This shows that

$$R_{jk|m|l} = a_{ml} R_{jk} + (1-n) b_{ml}\, g_{jk} + \tfrac{1}{4} c_{ml} W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l}. \quad (4.19)$$

If and only if

$$(R_{jk} - g_{jk}\, R)_{|m|l} = a_{ml}(R_{jk} - g_{jk}\, R). \quad (4.20)$$

In conclusion the proof of theorem is completed, we get

Theorem 4.5. In $G^{2nd}\, W_{|h} - BRF_n$, R-Ricci tensor R_{jk} is given in (4.19), if and only if the condition (4.20) is holds good.

Transvecting (4.18) by y^k, using (1.1a), (1.1c), (1.4b), (1.12e) and (1.13d), we get

$$R_{j|m|l} = a_{ml} R_j + (1-n) b_{ml}\, y_j + \tfrac{1}{3}\left(R_j - y_j R\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(R_j - y_j R\right). \quad (4.21)$$

This shows that

$$R_{j|m|l} = a_{ml} R_j + (1-n)\, b_{ml}\, y_j \, . \quad (4.22)$$

If and only if

$$\left(R_j - y_j R\right)_{|m|l} = a_{ml}\left(R_j - y_j R\right). \quad (4.23)$$

Transvecting (4.3) and (4.18) by g^{jk}, respectively using (1.2a), (1.2b), (1.12d), (1.13g), and (1.13h), we get

$$R^i_{h|m|l} = a_{ml} R^i_h + (n-1) b_{ml}\, \delta^i_h + \tfrac{1}{4}(n-1) c_{ml} W^i_h + \tfrac{1}{4}(n-1)\, \gamma_m\, W^i_{h|l} - \tfrac{1}{3}(n-1)\left(R^i_h\right)_{|m|l} - \tfrac{1}{3}(n-1) a_{ml} R^i_h \, . \quad (4.24)$$

This shows that

$$R^i_{h|m|l} = a_{ml} R^i_h + (n-1) b_{ml} \delta^i_h + \tfrac{1}{4}(n-1) c_{ml} W^i_h + \tfrac{1}{4}(n-1)\gamma_m\, W^i_{h|l}\, . \quad (4.25)$$

If and only if

$$\left(R^i_h\right)_{|m|l} = a_{ml} R^i_h \, . \quad (4.26)$$

And

$$R_{|m|l} = a_{ml} R + (1-n) n\, b_{ml} + \tfrac{1}{4} c_{ml} W + \tfrac{1}{4}\gamma_m W_{|l} + \tfrac{1}{3}(1-n)\left(R\right)_{|m|l} + \tfrac{1}{3}(1-n)\, a_{ml}\, R\, . \quad (4.27)$$

This shows that

$$R_{|m|l} = a_{ml} R + (1-n) n\, b_{ml} + \tfrac{1}{4} c_{ml} W + \tfrac{1}{4}\gamma_m W_{|l}\, . \quad (4.28)$$

If and only if

$$\left(R\right)_{|m|l} = a_{ml} R\, . \quad (4.29)$$

In conclusion the proof of theorem is completed, we get

Theorem 4.6. In $G^{2nd}\, W_{|h} - BRF_n$, vector R_j , the projective deviation tensor R^i_h and scalar R are given in (4.22), (4.25) and (4.28) if and only if the conditions (4.23), (4.26) and (4.29) are holds good, respectively.

Transvecting (4.3) by g_{ir}, using (1.1d), (1.2c), (1.12c) and (1.13f), we get

$$R_{rjkh|m|l} = a_{ml} R_{rjkh} + b_{ml}\left(g_{rk}\, g_{jh} - g_{rh}\, g_{jk}\right) + \tfrac{1}{3}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right) + \tfrac{1}{4} c_{ml}\left(W_{rk}\, g_{jh} - W_{rh}\, g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W_{rk}\, g_{jh} - W_{rh}\, g_{jk}\right)_{|l}\, . \quad (4.30)$$

This shows that

$$R_{rjkh|m|l} = a_{ml} R_{rjkh} + b_{ml}\left(g_{rk}\, g_{jh} - g_{rh}\, g_{jk}\right) + \tfrac{1}{4} c_{ml}\left(W_{rk}\, g_{jh} - W_{rh}\, g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W_{rk}\, g_{jh} - W_{rh}\, g_{jk}\right)_{|l}\, . \quad (4.31)$$

If and only if

$$\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} = a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right). \quad (4.32)$$

Thus, the proof of theorem is completed, we get

Theorem 4.7. In $G^{2nd}\, W_{|h} - BRF_n$, associate tensor R_{rjkh} (Cartan's 3th curvature tensor R^i_{jkh}) is a generalized birecurrent Finsler space if and only if the condition (4.32), holds good.

It is known that Cartan's 3th curvature tensor R^i_{jkh} and Cartan's 4th curvature tensor K^i_{jkh} are connected by the formula [17]

$$R^i_{jkh} = K^i_{jkh} + C^i_{jp}\, H^p_{kh}. \quad (4.33)$$

Taking the covariant derivative of (4.33), with respcet to x^m and x^l in the sence of Cartan we get

$$R^i_{jkh|m|l} = K^i_{jkh|m|l} + \left(C^i_{jp}\, H^p_{kh}\right)_{|m|l}. \quad (4.34)$$

Using (4.3) and (4.33) in (4.34) we get

$$K^i_{jkh|m|l} = a_{ml} K^i_{jkh} + b_{ml}\left(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\right) - \left(C^i_{jp}\, H^p_{kh}\right)_{|m|l} + a_{ml}\, C^i_{jp}\, H^p_{kh} + \tfrac{1}{4} c_{ml}\left(W^i_h\, g_{jk} - W^i_k\, g_{jh}\right) + \tfrac{1}{4}\gamma_m\left(W^i_h\, g_{jk} - W^i_k\, g_{jh}\right)_{|l} + \tfrac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right). \quad (4.35)$$

This shows that

$$K^i_{jkh|m|l} = a_{ml} K^i_{jkh} + b_{ml}\left(\delta^i_k\, g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{4} c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l}. \quad (4.36)$$

If and only if

$$\left(\delta_k^i R_{jh} - g_{jk} R_h^i\right)_{|m|l}$$
$$= a_{ml}\left(\delta_k^i R_{jh} - g_{jk} R_h^i\right). \quad (4.37)$$

and

$$\left(C_{jp}^i H_{kh}^p\right)_{|m|l} = a_{ml} C_{jp}^i H_{kh}^p . \quad (4.38)$$

Thus, the proof of theorem is completed, we get

Theorem 4.8. In $G^{2nd} W_{|h} - BRF_n$, Cartan's 4th curvature tensor K_{jkh}^i is a generalized birecurrent Finsler space if and only if the tensors $\left(\delta_k^i R_{jh} - g_{jk} R_h^i\right)$ and $\left(C_{jp}^i H_{kh}^p\right)$ are a generalized birecurrent Finsler space.

Contracting the indices i and h in the equations (4.35) and using (1.1d), (1.1b), (1.13e), (1.14e), (1.12c) and (1.12b), we get

$$K_{jk|m|l} = a_{ml}K_{jk} + (1-n)\, b_{ml}g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l} - \left(C_{jp}^i H_{ki}^p\right)_{|m|l} + a_{ml} C_{jp}^i H_{ki}^p + \frac{1}{3}\left(R_{jk} - g_{jk} R\right)_{|m|l} + \tfrac{1}{3}a_{ml}\left(R_{jk} - g_{jk} R\right). \quad (4.39)$$

This shows that

$$K_{jk|m|l} = a_{ml}K_{jk} + (1-n)b_{ml}\, g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l} . \quad (4.40)$$

If and only if

$$\left(R_{jk} - g_{jk} R\right)_{|m|l} = a_{ml}\left(R_{jk} - g_{jk} R\right). \quad (4.41)$$

And $\left(C_{jp}^i H_{ki}^p\right)_{|m|l} = a_{ml} C_{jp}^i H_{kh}^p$. (4.42)

Transvecting (4.39) by y^k, using (1.1a), (1.1c), (1.4b), (1.12a), (1.12e), (1.14c) and (1.13d), we get

$$K_{j|m|l} = a_{ml}K_j + (1-n)b_{ml}\, y_j + \frac{1}{3}\left(R_j - y_j R\right)_{|m|l} + \tfrac{1}{3}a_{ml}\left(R_j - y_j R\right) - \left(C_{jp}^i H_i^p\right)_{|m|l} + a_{ml} C_{jp}^i H_i^p . \quad (4.43)$$

This shows that

$$K_{j|m|l} = a_{ml}K_j + (1-n)\, b_{ml}y_j . \quad (4.44)$$

If and only if

$$\left(R_j - y_j R\right)_{|m|l} = a_{ml}\left(R_j - y_j R\right). \quad (4.45)$$

and

$$\left(C_{jp}^i H_i^p\right)_{|m|l} = a_{ml} C_{jp}^i H_i^p . \quad (4.46)$$

Thus, the proof of theorem is completed, we get

Theorem 4.9. In $G^{2nd} W_{|h} - BRF_n$, K−Ricci tensor K_{jk} and curvature vector K_j is given in (4.40) and (4.44), if and only if the conditions (4.41), (4.42), (4.45) and (4.46) are holds good, respectively.

Transvecting (4.35) and (4.39) by g^{jk}, respectively using the equations (1.2a), (1.2b), (1.12d), (1.14d), and (1.14f), we get

$$K_{h|m|l}^i = a_{ml}K_h^i + (n-1)b_{ml}\, \delta_h^i + \frac{1}{4}(n-1)c_{ml}\, W_h^i + \tfrac{1}{4}(n-1)\gamma_m W_{h|l}^i + \tfrac{1}{3}(1-n)\left(R_h^i\right)_{|m|l} + \tfrac{1}{3}(1-n)a_{ml} R_h^i - g^{jk}\left(C_{jp}^i H_{kh}^p\right)_{|m|l} + a_{ml}g^{jk}\left(C_{jp}^i H_{kh}^p\right). \quad (4.47)$$

This shows that

$$K_{h|m|l}^i = a_{ml}K_h^i + (n-1)b_{ml}\, \delta_h^i + \tfrac{1}{4}(n-1)\, c_{ml}\, W_h^i + \tfrac{1}{4}(n-1)\, \gamma_m W_{h|l}^i . \quad (4.48)$$

If and only if

$$\left(C_{jp}^i H_{kh}^p\right)_{|m|l} = a_{ml} C_{jp}^i H_{kh}^p , \text{ where } g^{jk} \neq 0 . \quad (4.49)$$

And

$$\left(R_h^i\right)_{|m|l} = a_{ml} R_h^i . \quad (4.50)$$

Also,

$$K_{|m|l} = a_{ml}K + (1-n)n\, b_{ml} + \tfrac{1}{4}c_{ml}W + \tfrac{1}{4}\gamma_m W_{|l} + \tfrac{1}{3}(1-n)(R)_{|m|l} + \tfrac{1}{3}(1-n)a_{ml}R - g^{jk}\left(C_{jp}^i H_{kh}^p\right)_{|m|l} + a_{ml}\, g^{jk}\left(C_{jp}^i H_{kh}^p\right) . \quad (4.51)$$

This shows that

$$K_{|m|l} = a_{ml}K + (1-n)n\, b_{ml} + \tfrac{1}{4}c_{ml}W + \tfrac{1}{4}\gamma_m W_{|l} . \quad (4.52)$$

If and only if

$$\left(C_{jp}^i H_{kh}^p\right)_{|m|l} = a_{ml}\left(C_{jp}^i H_{kh}^p\right), \text{ where } g^{jk} \neq 0 , \quad (4.53)$$

And $(R)_{|m|l} = a_{ml}R$. (4.54)

Thus, the proof of theorem is completed, we get

Theorem 4.10. In $G^{2nd} W_{|h} - BRF_n$, the projective deviation tensor K_h^i and scalar K are given in (4.48) and (4.52) if and only if the conditions (4.49), (4.50), (4.53) and (4.54) are holds good, respectively.

Conclusions

In this paper, we have extensively studied the generalized $W_{|h}$ -recurrent Finsler space, denoted by $G^{2nd} W_{|h} - BRF_n$, and its associated curvature and torsion tensors. Our analysis has led to the formulation of various conditions and theorems that describe the behavior of these tensors under specific transformations. We derived the essential equations governing the covariant derivatives of the Wely's projective curvature tensor in the context of higher-dimensional spaces, illustrating the non-trivial interactions between the curvature and torsion tensors. Specifically, we showed that the space $G^{2nd}\,W_{|h} - BRF_n$ exhibits generalized birecurrent behavior, a property that links the torsion and deviation tensors in a consistent mathematical framework. Additionally, we demonstrated that the Ricci tensor, the deviation tensor, and the torsion tensor exhibit specific conditions under which they maintain their generalized birecurrent Finsler space characteristics. Our results also clarify how these tensors transform when subjected to higher-dimensional transvection conditions, leading to the establishment of key relationships and identities between them.

The results presented in this paper are significant for advancing the understanding of generalized Finsler spaces, particularly those characterized by recurrent or birecurrent properties. The derived conditions offer valuable insights for future research in geometric structures and their applications in various fields, such as differential geometry and theoretical physics.

Moreover, the theorems established, particularly Theorems 3.1, 3.2, 4.1, and 4.5, provide a comprehensive framework for studying the curvature and torsion tensors in generalized Finsler spaces. These findings can contribute to the development of more sophisticated models for geometric objects in higher-dimensional spaces, with potential applications in areas such as general relativity and cosmology.

Future work should focus on further exploring the implications of these results in practical contexts, including their relationship with other geometric structures and their potential role in the study of spacetime models. Additionally, investigating the stability of these generalized Finsler spaces under various transformations could provide further insights into the underlying geometric properties.

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مقالة بحثية

تعميم موتر انحناء وِليّ الإسقاطي في فضاءات فينسلر: دراسة حول موترات الــW احادي الاشتقاق، ثنائي الاشتقاق، وموترات ريتشي

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المُلخّص

في هذه الدراسة، نستكشف تعميم موتر انحناء وِليّ الإسقاطي W^i_{jkh} الذي يحقق علاقة اشتقاقية محددة في (3.1)، والتي تعرف فضاءً عامًا يُسمى فضاءً $W_{|h} - BRF_n$ $G^{2nd}-$ الاشتقاقي. من خلال دراسة الشروط التي يحققها هذا الموتر في علاقات الاشتقاقات المتكررة العامة، نقوم بتعريف وتحليل خصائص الموترات-W للاشتقاق الاحادي، والاشتقاق الثنائي، وموتر ريتشي في سياق المشتقات المتغيرة من الدرجة الثانية. نستنتج سلسلة من المعادلات التي تصف سلوك هذه الموترات، بما في ذلك تعبيرات المشتق المتغيير وعلاقاتها مع انحناءات القياس وموترات الانحراف. تتمثل المساهمة الرئيسية في إثبات العديد من النظريات التي تربط شروط التكرارات العامة بموترات الالتواء وخصائص الانحناء. على وجه التحديد، نثبت أن موتر ريتشي والموترات المرتبطة به تُظهر سلوك فضاء فينسلر ثنائي الاشتقاق تحت ظروف معينة. تقدم النتائج رؤى إضافية حول موترات الالتواء والانحراف، مما يبرز دورها في هيكل الانحناء العام لفضاءات فينسلر. هذه النتائج توسع نظرية الفضاءات الراجعة، مقدمة إطارًا شاملاً لفهم الظواهر الهندسية للانحناء من الدرجات العليا في هندسة فينسلر.

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