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RESEARCH ARTICLE

GENERALIZED H^h-RECURRENT FINSLER GEOMETRY WITH APPLICATIONS TO ANISOTROPIC IMAGE PROCESSING

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Abstract

In this paper, we investigate the structure of generalized H^h-recurrent Finsler spaces (G-H^h-R-F_n) and establish several recurrence relations for Cartan's h-curvature tensor and its associated geometric invariants. In particular, Theorems 3.1, 3.2, and 4a.3 provide novel conditions characterizing the stability and recurrence of curvature under horizontal covariant differentiation. To demonstrate the practical significance of these results, we extend the theoretical framework to the domain of digital image processing. A Finslerian metric derived from image gradients is constructed to model anisotropic features, and the recurrence conditions are shown to enhance edge preservation during anisotropic diffusion filtering. Simulation steps are outlined, illustrating how the recurrence properties of curvature tensors improve noise suppression and directional stability compared to standard Euclidean methods. The proposed approach highlights the dual role of generalized Finsler recurrence: as a fundamental extension in differential geometry and as a powerful tool for advanced computer vision applications such as denoising, segmentation, and texture analysis.

Keywords: Finsler geometry, H^h-recurrence, Cartan curvature tensor, Anisotropy, Digital image processing, Anisotropic diffusion.

1. Introduction

Finsler geometry continues to be a rich field of research due to its capacity to generalize Riemannian geometry and its applicability in both theoretical and applied sciences. The work in [1] introduced higher-order Cartan derivatives and curvature tensors, offering a deeper understanding of differential structures in Finsler spaces. A global minimal path framework using Finsler elastica models was proposed in [2], providing a robust approach for solving variational problems, particularly in image analysis. This direction was extended in [3] through fast asymmetric front propagation methods for image segmentation tasks, enhancing computational efficiency.

Geometric diffusion processes within Finsler and Riemannian manifolds were comprehensively analyzed in [4], offering insights into the interplay between geometric structures and diffusion behavior. In parallel, [5] investigated generalized harmonic maps with practical implications in image processing, showcasing the relevance of Finsler geometry in modern imaging techniques. The use of transformer-based deep learning models for image restoration was exemplified in [6], marking a shift toward data-driven approaches in image processing.

The concept of generalized h-recurrent Finsler connections was addressed in [7] and expanded through foundational geometric identities in [8], both of which contribute to the structural understanding of Finsler spaces. The implementation of level set methods for anisotropic geometric diffusion in 3D imaging was introduced in [9], integrating geometric modeling with computational methods. In [10], a Finsler metric-based method was utilized for ship detection in SAR images, reinforcing the applicability of Finsler models in remote sensing and target recognition.

Advanced techniques in hyperspectral image denoising and inpainting were proposed in [11], leveraging low-rank and sparse representations to improve data reconstruction quality. Further theoretical advancements in recurrent tensor field decomposition within Finsler spaces were detailed in [12], while identities in generalized R^h-recurrent spaces were formulated in [13]. These developments were reinforced by the additional materials and alternate publishing platforms provided in [14-20], ensuring accessibility and reproducibility of the research.

A foundational perspective on Finsler spaces and their generalizations was presented in [21], establishing a rigorous mathematical basis for further developments. Lastly, the pullback formalism and connections in Finsler geometry were discussed in [22], providing essential tools for advanced geometric modeling.

2. Preliminaries

Let F_n be an nnn-dimensional Finsler space endowed with the metric function F satisfying the standard regularity conditions. The components of the associated metric tensor g_{ij} , Cartan's connection coefficients Γ^{*i}_{jk} , and Berwald's connection coefficients G^i_{jk} are symmetric in their lower indices and positively homogeneous of degree zero with respect to the directional arguments. The contravariant and covariant metric tensors satisfy the standard relation:

(2.1) $$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \ . \end{cases}$$

where δ^i_j is the Kronecker delta.

The tangent vectors y_i and y^i satisfy the following conditions:

(2.2) a) $y_i = g_{ij}\, y^j$,

b) $y_i y^i = F^2$,

c) $g_{ij} = \dot{\partial}_i\, y_j = \dot{\partial}_j\, y_i$,

d) $g_{ij}\, y^j = \frac{1}{2}\dot{\partial}_i F^2 = F\dot{\partial}_i F$ and

e) $\dot{\partial}_j\, y^i = \delta^i_j$.

The (h)hv-torsion tensor C_{ijk} is defined as

(2.3) $$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 \ ,$$

which is symmetric in all indices and positively homogeneous of degree -1 in the directional argument. Its associated (v)hv-torsion tensor C^h_{jk} is given by

(2.4) a) $C^h_{ik} = g^{hj}\, C_{ijk}$ and

b) $C_{ijk} = g_{hj}\, C^h_{ik}$.

And shares the same symmetry and homogeneity properties.

The unit vector along the direction of y^i is

(2.5) a) $l^i = \frac{y^i}{F}$ and

b) $l_i = g_{ij} l^j = \dot{\partial}_i\, F = \frac{y_i}{F}$.

For an arbitrary vector field X^i , the h-covariant derivative with respect to x^k is defined by Cartan as:

(2.6) $$X^i_{|k} = \partial_k\, X^i - \left(\dot{\partial}_r\, X^i\right) G^r_k + X^r \Gamma^{*i}_{rk} \ .$$

Where the functions Γ^{*i}_{rk} and G^r_k are defined by

(2.7) a) $\Gamma^{*i}_{rk} = \Gamma^i_{rk} - C^i_{mr}\, \Gamma^m_{sk}\, y^s$ and

b) $G^r_k = \Gamma^{*r}_{sk}\, y^s$.

The h-covariant differentiation preserves the metric and tangent vector, i.e.,

(2.8) a) $g_{ij|k} = 0$ and

b) $y^i_{|k} = 0$.

and commutes with differentiation with respect to y^j according to

(2.9) $$\dot{\partial}_j \left(X^i_{|k}\right) - \left(\dot{\partial}_j X^i\right)_{|k} = X^r \left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk} \ ,$$

Also, we have

(2.10) a) $P^i_{jkh}\, y^j = \Gamma^{*i}_{kjh}\, y^j = P^i_{kh} = C^i_{kh|r}\, y^r$,

b) $\dot{\partial}_j \Gamma^{*r}_{hk} = \Gamma^{*r}_{hkj}$,

c) $P^i_{kh}\, y^k = 0 = P^i_{kh}\, y^h$, and

d) $y_i\, \Gamma^{*i}_{kjh} = -P_{kjh}$.

Where P^i_{jkh} is the (v)hv-torsion tensor, with its associated form

(2.11) $$g_{rj}\, P^r_{kh} = P_{kjh} \ .$$

In Landsberg spaces, the Berwald connection coincides with Cartan's connection, satisfying

(2.12) $$y_r\, G^r_{jkh} = -2C_{ijk|s}\, y^s = -2\, P_{jkh} = 0 \ .$$

Various authors denote the tensor $C_{jkh|s}\, y^s$ by P_{jkh} .

The h-curvature (Berwald curvature) and h(v)-torsion tensors are defined by

(2.13) a) $H^i_{jkh} = \partial_j\, G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj}\, G^r_k$

$-\partial_j\, G^i_{hk} - G^r_{hk}\, G^i_{rj} - G^i_{rkj}\, G^r_h$, and

b) $H^i_{kh} = \partial_h G^i_k + G^r_k\, C^i_{rh} - \partial_k G^i_h - G^r_h\, C^i_{rk}$,

and satisfy standard homogeneity, symmetry, and contraction relations, leading to the Ricci tensor, curvature vector, and scalar curvature:

(2.14) a) $H^i_{jkh}\, y^j = H^i_{kh}$,

b) $H^i_{jkh} = \partial_j\, H^i_{kh}$, and

c) $H^i_{jk} = \partial_j\, H^i_k$.

These tensors were constructed initially by mean of the tensor H_h^i , called the deviation tensor, given by

(2.15) $H_h^i = 2\partial_h G^i - \partial_r G_h^i\, y^r + 2G_{hs}^i\, G^s - G_s^i\, G_h^s$.

The deviation tensor H_h^i is positively homogeneous of degree two with respect to the directional argument. Applying Euler's theorem for homogeneous functions and performing the contraction of the indices i and h in equations (2.14) and (2.15), the following relations are obtained:

(2.16) a) $H_{jk}^i\, y^j = -H_{kj}^i\, y^j = H_k^i$,

b) $H_{jk} = H_{jkr}^r$,

c) $H_j = H_{jr}^r$,

d) $(n-1)H = H_r^r$,

e) $H_{rkh}^r = H_{hk} - H_{kh}$, and

f) $y_i H_j^i = 0$.

These relations describe the contraction properties of the deviation tensor and establish the connections between the Ricci tensor, curvature vector, and scalar curvature within the Finsler geometric framework. The contracted tensor H_{kh} (Ricci tensor), the curvature vector H_k , and the scalar curvature H are interrelated through the following relations, reflecting the intrinsic geometric properties of the Finsler space.

(2.17) a) $H_{kh} = \dot{\partial}_k H_h$,

b) $H_{kh}\, y^k = H_h$ and

c) $H_k\, y^k = (n-1)\, H$.

The tensors H_{jkh}^i and H_{kh}^i satisfy the subsequent identities, which ensure consistency within the h-curvature framework.

(2.18) a) $H_{ijkh} = g_{jr}\, H_{ihk}^r$,

b) $H_{jk.h} = g_{jr} H_{hk}^r$,

c) $y_i H_{hk}^i = 0$,

d) $H_{jkh}^i + H_{hjk}^i + H_{khj}^i = 0$.

Furthermore, the necessary and sufficient condition for an n-dimensional Finsler space $F_n (n > 2)$ to possess scalar curvature is expressed as:

(2.19) $H_h^i = F^2\, R\big(\delta_h^i - l^i l_h\big)$,

where R denotes the scalar curvature, δ_h^i is the Kronecker delta, and l^i represents the unit vector in the direction of the tangent to the Finsler manifold. This condition provides a precise criterion for identifying scalar-curvature Finsler spaces within the generalized geometric setting.

Finally, the third Cartan curvature tensor R_{jkh}^i obeys the Bianchi identities and relates to the Berwald curvature via

(2.20) $R_{jkh|s}^i + R_{jsk|h}^i + R_{jhs|k}^i$

$+\big(R_{mhs}^r P_{jkr}^i + R_{mkh}^r P_{jsr}^i + R_{msk}^r P_{jhr}^i\big) y^m = 0$.

and this tensor satisfies the following relation too

(2.21) a) $R_{jkh}^i = K_{jkh}^i + C_{js}^i\, K_{rhk}^s\, y^r$, and

b) $R_{ijkh} = K_{ijkh} + C_{ijs}\, H_{kh}^s$.

The curvature tensor K_{jkh}^i satisfies the following known as Bianchi identities

(2.22) $K_{jkh}^i + K_{hjk}^i + K_{khj}^i = 0$, and

(2.23) $K_{jrkh} + K_{hrjk} + K_{krhj} = 0$

3. Necessary and Sufficient Condition for Generalized h-Recurrent

In the study of Finsler geometry, recurrent structures provide a powerful framework for characterizing the behavior of curvature tensors under covariant differentiation. Among these, generalized recurrent conditions play a central role in extending classical results to broader settings. In particular, the concept of a generalized H^h-recurrent Finsler space arises when the Berwald curvature tensor satisfies a specific linear relation involving two non-null covariant vector fields. Such a structure, denoted by $G\text{-}H^h\text{-}R\text{-}F_n$, enables the systematic analysis of the interplay between curvature tensors, torsion tensors, and their associated contractions.

This section develops the necessary and sufficient conditions for a Finsler space to be generalized H^h-recurrent. By employing transvection operations and utilizing fundamental identities from earlier sections, we establish explicit relations for the covariant derivatives of the Berwald curvature tensor and its associated tensors. Furthermore, a sequence of theorems is presented, characterizing the precise conditions under which these tensors preserve h-recurrence. These results not only generalize existing recurrence conditions in Finsler geometry but also provide new insights into the structural properties of higher-order curvature tensors.

Consider an n-dimensional Finsler space F_n whose Berwald curvature tensor H_{jkh}^i satisfies the relation

(3.1) $H_{jkh|l}^i = \lambda_l H_{jkh}^i + \mu_l\big(\delta_h^i\, g_{jk} - \delta_k^i\, g_{jh}\big)$

$+\frac{1}{4}\delta_l\big(H_h^i\, g_{jk} - H_k^i\, g_{jh}\big)$, $H_{jkh}^i \neq 0$,

where λ_l , μ_l and δ_l are non-vanishing covariant vector fields. A Finsler space fulfilling this condition is referred to as a generalized H^h-recurrent Finsler space, denoted by $G\text{-}H^h\text{-}R\text{-}F_n$.

By contracting and transvecting this defining condition with suitable tensorial components, several equivalent forms of recurrence relations are obtained. These relations characterize the behavior of the Berwald curvature tensor, its associated torsion tensor, and the deviation tensor under h-covariant differentiation. As a result, a series of theorems are established, each providing the necessary and sufficient condition for these tensors to preserve the h-recurrence property.

In particular, the analysis demonstrates how the recurrence of the Berwald curvature tensor and its associated tensors depends on the interaction between the covariant vectors λ_l , μ_l and δ_l and the underlying Finsler metric. The derived conditions not only generalize existing recurrence structures but also highlight new connections between higher-order curvature tensors within the broader framework of Finsler geometry.

Now, let us consider a generalized H^h-recurrent Finsler space defined by the condition (3.1).

By transvecting equation (3.1) with y^j, and making use of relations (2.8b), (2.14a), and (2.2a), we obtain a new form of the recurrence relation.

(3.2) $$H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l(\delta^i_h\, y_k - \delta^i_k\, y_h) + \tfrac{1}{4}\delta_l(H^i_h\, y_k - H^i_k\, y_h) \ .$$

Furthermore, transvecting equation (3.2) with y^k, while applying (2.8b), (2.16a), (2.2b), and (2.1), leads to an additional expression.

(3.3) $$H^i_{h|l} = \lambda_l H^i_h + \mu_l(\delta^i_h F^2 - y_h\, y^i) + \tfrac{1}{4}\delta_l(H^i_h F^2 - H^i_k\, y_h\, y^k)\, .$$

Similarly, transvecting equation (3.2) with g_{ip}, and employing (2.18b), (2.8a), and (2.1), we arrive at another equivalent condition.

(3.4) $$H_{kp.h|l} = \lambda_l H_{kp.h} + \mu_l(g_{hp}\, y_k - g_{kp}\, y_h) + \tfrac{1}{4}\delta_l\, g_{ip}(H^i_h F^2 - H^i_k\, y_h\, y^k) \ .$$

Accordingly, we establish the following:

Theorem 3.1. The h-covariant derivatives of the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h in a generalized H^h-recurrent Finsler space G-H^h-R-F_n are expressed through conditions (3.2), (3.3), and (3.4), respectively.

By differentiating equation (3.2) partially with respect to y^j, and applying relations (2.14b), (2.14c), (2.2c), and (2.10b), together with the commutation formula given in (2.9) for the h(v)-torsion tensor H^i_{jk}, we obtain the following result:

(3.5)
$$\begin{aligned} H^i_{jkh|l} + H^r_{kh}\Gamma^{*i}_{jrl} - H^i_{rh}\Gamma^{*r}_{jkl} - H^i_{kr}\Gamma^{*r}_{jhl} - H^i_{rkh}P^r_{jl} &= (\dot\partial_j\lambda_l)H^i_{kh} + \lambda_l H^i_{jkh} \\ &+ (\dot\partial_j\mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) \\ &+ \mu_l\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \\ &+ \frac{1}{4}(\dot\partial_j\delta_l)(H^i_h\, y_k - H^i_k\, y_h) \\ &+ \frac{1}{4}\delta_l(H^i_{jh}\, y_k - H^i_{jk}\, y_h). \end{aligned}$$

This shows that

(3.6) $$H^i_{jkh|l} = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) + \tfrac{1}{4}\delta_l(H^i_{jh}\, y_k - H^i_{jk}\, y_h) \ .$$

If and only if

(3.7) $$\begin{aligned} H^r_{kh}\Gamma^{*i}_{jrl} - H^i_{rh}\Gamma^{*r}_{jkl} - H^i_{kr}\Gamma^{*r}_{jhl} - H^i_{rkh}P^r_{jl} &= (\dot\partial_j\lambda_l)H^i_{kh} + (\dot\partial_j\mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) \\ &+ \tfrac{1}{4}(\dot\partial_j\delta_l)(H^i_h\, y_k - H^i_k\, y_h) \ . \end{aligned}$$

Thus, we conclude

Theorem 3.2. In a G-H^h-R-F_n, the Berwald curvature tensor H^i_{jkh} is h-recurrent provided that condition (3.7) is satisfied.

By transvecting condition (3.5) with g_{ip} , and making use of (2.8a), (2.18a), and (2.1), we arrive at the following:

(3.8)
$$\begin{aligned} &H_{jpkh|l} + g_{ip}\,[H^r_{kh}\Gamma^{*i}_{jrl} - H^i_{rh}\Gamma^{*r}_{jkl} - H^i_{kr}\,\Gamma^{*r}_{jhl} - H^i_{rkh}P^r_{jl}] \\ &= \lambda_l H_{jpkh} + g_{ip}\,[(\dot\partial_j\lambda_l)H^i_{kh} + (\dot\partial_j\mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h)] \\ &\quad + \mu_l(g_{jk}\, g_{hp} - g_{jh}\, g_{kp}) \\ &+ \tfrac{1}{4}(\dot\partial_j\delta_l)(H^i_h\, y_k - H^i_k\, y_h)\, g_{ip} + \tfrac{1}{4}\delta_l(H^i_{jh}\, y_k - H^i_{jk}\, y_h)\, g_{ip} \end{aligned}$$

This shows that

(3.9) $$H_{jpkh|l} = \lambda_l\, H_{jpkh} + \mu_l(g_{jk}\, g_{hp} - g_{jh}\, g_{kp}) + \tfrac{1}{4}\delta_l(H^i_{jh}\, y_k - H^i_{jk}\, y_h)\, g_{ip} \ .$$

If and only if

(3.10) $$\begin{aligned} H^r_{kh}\Gamma^{*i}_{jrl} - H^i_{rh}\Gamma^{*r}_{jkl} - H^i_{kr}\,\Gamma^{*r}_{jhl} - H^i_{rkh}\,P^r_{jl} &= (\dot\partial_j\lambda_l)H^i_{kh} + (\dot\partial_j\mu_l)(\delta^i_h\, y_k - \delta^i_k\, y_h) \\ &+ \tfrac{1}{4}(\dot\partial_j\delta_l)(H^i_h\, y_k - H^i_k\, y_h). \end{aligned}$$

Thus, we conclude

Theorem 3.3. In a G-H^h-R-F_n , the associate tensor H_{jpkh} of the Berwald curvature tensor H^i_{jkh} is h-recurrent if and only if condition (3.10) holds.

By contracting the indices i and h in condition (3.5), and using relations (2.16b), (2.16d), (2.16f), (2.16c) and (2.18c), we obtain:

(3.11) $$H_{jk|l} + H^r_{kp}\Gamma^{*p}_{jrl} - H_r\Gamma^{*r}_{jkl} - H^p_{kr}\Gamma^{*r}_{jpl} - H_{rk}P^r_{jl} = \lambda_l H_{jk} + (\dot{\partial}_j\lambda_l)H_k + (n-1)(\dot{\partial}_j\mu_l)y_k + (n-1)\mu_l g_{jk} + \tfrac{1}{4}(n-1)(\dot{\partial}_j\delta_l)(Hy_k) + \tfrac{1}{4}\delta_l(H_j y_k)\,.$$

This shows that

(3.12) $$H_{jk|l} = \lambda_l H_{jk} + (n-1)\,\mu_l g_{jk} + \tfrac{1}{4}\delta_l(H_j y_k)\ .$$

If and only if

(3.13) $$H^r_{kp}\Gamma^{*p}_{jrl} - H_r\Gamma^{*r}_{jkl} - H^p_{kr}\Gamma^{*r}_{jpl} - H_{rk}P^r_{jl} = (\dot{\partial}_j\lambda_l)H_k + (n-1)(\dot{\partial}_j\mu_l)y_k + \tfrac{1}{4}(n-1)(\dot{\partial}_j\delta_l)(Hy_k)\,.$$

Thus, we conclude

Theorem 3.4. In a G-H^h-R-F_n , the associate tensor H_{jk} of the Berwald curvature tensor H^i_{jkh} is h-recurrent if and only if condition (3.13) is satisfied.

By contracting the indices i and j in equation (3.5), and applying (2.16e), (2.16c) and (2.1), we derive the following:

(3.14) $$(H_{hk} - H_{kh})_{|l} + H^r_{kh}\Gamma^{*p}_{prl} - H^p_{rh}\Gamma^{*r}_{pkl} - H^p_{kr}\Gamma^{*r}_{phl} - H^p_{rkh}P^r_{pl} = \lambda_l(H_{hk} - H_{kh}) + (\dot{\partial}_p\lambda_l)\,H^p_{kh} + (\dot{\partial}_p\mu_l)(\delta^p_h y_k - \delta^p_k y_h) + \tfrac{1}{4}(\dot{\partial}_p\delta_l)(H^p_h y_k - H^p_k y_h) + \tfrac{1}{4}\delta_l(H_h y_k - H_k y_h).$$

This shows that

(3.15) $$(H_{hk} - H_{kh})_{|l} = \lambda_l(H_{hk} - H_{kh})\ .$$

If and only if

(3.16) $$H^r_{kh}\,\Gamma^{*p}_{prl} - H^p_{rh}\,\Gamma^{*r}_{pkl} - H^p_{kr}\,\Gamma^{*r}_{phl} - H^p_{rkh}\,P^r_{pl} = (\dot{\partial}_p\lambda_l)\,H^p_{kh} + (\dot{\partial}_p\,\mu_l)(\delta^p_h\,y_k - \delta^p_k\,y_h) + \tfrac{1}{4}(\dot{\partial}_p\delta_l)(H^p_h y_k - H^p_k y_h) + \tfrac{1}{4}\delta_l(H_h y_k - H_k y_h).$$

Thus, we conclude

Theorem 3.5. In a G-H^h-R-F_n , the associate tensor $(H_{hk} - H_{kh})$ of the Berwald curvature tensor H^i_{jkh} is h-recurrent if and only if condition (3.16) holds.

Next, by differentiating equation (3.3) partially with respect to y^k, and applying (2.14c), (2.2d), (2.2a), (2.2c), and (2.2e), together with the commutation formula given in (2.9) for the h(v)-torsion tensor H^i_h, we obtain an additional relation.

(3.17) $$H^i_{kh|l} + H^r_h\,\Gamma^{*i}_{krl} - H^i_r\,\Gamma^{*r}_{khl} - H^i_{rh}\,P^r_{kl} = (\dot{\partial}_k\lambda_l)H^i_h + \lambda_l H^i_{kh} + (\dot{\partial}_k\mu_l)(\delta^i_h F^2 - y_h\,y^i) + \mu_l\,(2\delta^i_h\,y_k - g_{kh}y^{\,i} - \delta^i_k\,y_h) + \tfrac{1}{4}(\dot{\partial}_k\delta_l)(H^i_h F^2 - H^i_k\,y_h\,y^k) + \tfrac{1}{4}\delta_l(H^i_{hk}F^2 - H^i_k\,y_h)\,.$$

Furthermore, interchanging the indices k and h in condition (3.17), and subtracting the resulting equation from (3.17), yields a refined expression that further characterizes the recurrence structure.

(3.18)
$$\left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\right)_{|l} + \left[\,H^r_h\Gamma^{*i}_{krl} - H^i_r\,\Gamma^{*r}_{khl} - H^i_{rh}P^r_{kl} - H^r_k\Gamma^{*i}_{hrl} + H^i_r\,\Gamma^{*r}_{hkl} + H^i_{rk}\,P^r_{hl}\right] = \lambda_l\left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\right) + 3\,\mu_l(\delta^i_h\,y_k - \delta^i_k\,y_h) + \left[(\dot{\partial}_k\lambda_l)H^i_h + (\dot{\partial}_k\mu_l)(\delta^i_h F^2 - y_h y^i) - (\dot{\partial}_h\lambda_l)H^i_k - (\dot{\partial}_h\mu_l)(\delta^i_k\,F^2 - y_k\,y^i)\right] + \frac{1}{4}(\dot{\partial}_k\delta_l)(H^i_h F^2 - H^i_k\,y_h\,y^k) + \frac{1}{4}\delta_l(H^i_{hk}F^2 - H^i_k\,y_h).$$

4. Special Generalized H^h-Recurrent Finsler Spaces

In this section, we investigate several special cases of generalized H^h-recurrent Finsler spaces that play an essential role in the structural study of Finsler geometry. We begin with the case of affinely connected spaces (Berwald spaces), where the connection parameters become independent of the directional arguments, leading to simplified recurrence relations for the Berwald curvature tensor and its associated tensors. Subsequently, we consider P2-like generalized H^h-recurrent spaces, which are characterized by specific relations involving the torsion tensors and curvature identities. Finally, we address the P*-generalized H^h-recurrent spaces, highlighting their connections with P2-like spaces and the recurrence conditions imposed on their torsion tensors. These subclasses provide deeper insights into the geometry of generalized recurrent Finsler spaces and demonstrate how particular structural assumptions lead to distinct forms of curvature recurrence.

4a. A Generalized H^h-Recurrent Affinely Connected Space

A Finsler space F_n in which the connection coefficients G^i_{jk} are independent of the directional arguments y^i is referred to as an affinely connected space or Berwald space. Consequently, an affinely connected Finsler space can be equivalently characterized by any of the following conditions:

(4a.1) a) $G^i_{jkh} = 0$ and b) $C_{(ijk|h)} = 0$.

In such a space, the connection parameters of Cartan, Γ^{*i}_{kh}, coincide with the Berwald connection coefficients G^i_{kh} and are independent of the directional arguments, i.e.,

(4a.2) a) $\dot{\partial}_j G^i_{kh} = 0$ and b) $\dot{\partial}_j\, \Gamma^{*i}_{kh} = 0$.

Definition 4.1. A generalized H^h-recurrent Finsler space FnF_nFn is termed an affinely connected generalized H^h-recurrent Finsler space if it satisfies at least one of the conditions (4a.1a), (4a.1b), (4a.2a), or (4a.2b). Such a space is denoted concisely as $G\text{-}H^h\text{-}R\text{-}F_n$ affinely connected space.

Now, consider a $G\text{-}H^h\text{-}R\text{-}F_n$ space that is affinely connected. If the directional derivatives of the covariant vector fields vanish, i.e., $\dot{\partial}_j\lambda_l = 0$, $\dot{\partial}_j\mu_l = 0$ and $\dot{\partial}_j\delta_l = 0$, then in view of condition (4a.2b), equation (3.5) reduces to:

(4a.3) $$H^i_{jkh|l} = \lambda_l H^i_{jkh} + H^i_{rkh}\, P^r_{jl} + \mu_l\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\delta_l\left(H^i_{jh}\, y_k - H^i_{jk}\, y_h\right).$$

Thus, we conclude

Theorem 4a.1. In a $G\text{-}H^h\text{-}R\text{-}F_n$ affinely connected space, if the directional derivatives of the first-order covariant tensor fields vanish, then the Berwald curvature tensor H^i_{jkh} is h-recurrent.

Assume now that a $G\text{-}H^h\text{-}R\text{-}F_n$ space is affinely connected and that $\dot{\partial}_j\lambda_l = 0$, $\dot{\partial}_j\mu_l = 0$ and

$\dot{\partial}_j\delta_l = 0$. Under these conditions, in view of (4a.2b) and using (2.18a), equation (3.8) reduces to

(4a.4) $$H_{jpkh|l} = \lambda_l H_{jpkh} + H_{rpkh} P^r_{jl} + \mu_l\left(g_{jk}g_{hp} - g_{jh}g_{kp}\right) + \tfrac{1}{4}\delta_l\left(H^i_{jh} y_k - H^i_{jk} y_h\right) g_{ip}\,.$$

Thus, we conclude

Theorem 4a.2. In such a space, if the directional derivatives of the first-order covariant tensor fields vanish, then the associated tensor H_{jpkh} is h-recurrent.

Similarly, for the same affinely connected space with $\dot{\partial}_j\lambda_l = 0$, $\dot{\partial}_j\mu_l = 0$ and $\dot{\partial}_j\delta_l = 0$, equation (3.11) reduces to

(4a.5) $$H_{jk|l} = \lambda_l H_{jk} + H_{rk} P^r_{jl} + (n-1)\,\mu_l\, g_{jk} + \tfrac{1}{4}\delta_l\left(H_j y_k\right) .$$

Thus, we conclude

Theorem 4a.3. Under these conditions, the Ricci tensor H_{jk} of the Berwald curvature tensor H^i_{jkh} is h-recurrent.

Considering the affinely connected space $G\text{-}H^h\text{-}R\text{-}F_n$ and assuming $\dot{\partial}_j\lambda_l = 0$, $\dot{\partial}_j\mu_l = 0$ and $\dot{\partial}_j\delta_l = 0$, equation (3.14) reduces to

(4a.6) $$(H_{hk} - H_{kh})_{|l} = \lambda_l(H_{hk} - H_{kh}) + H^p_{rkh}\, P^r_{pl} + \tfrac{1}{4}\delta_l(H_h y_k - H_k y_h) .$$

Thus, we conclude

Theorem 4a.4. In this case, the tensor $(H_{hk} - H_{kh})$ is h-recurrent.

Similarly, under the same assumptions, equation (3.17) reduces to

(4a.7) $$H^i_{kh|l} = \lambda_l H^i_{kh} + H^i_{rh}\, P^r_{kl} + \mu_l\left(2\delta^i_h\, y_k - g_{kh}\, y^i - \delta^i_k y_h\right) + \tfrac{1}{4}\delta_l\left(H^i_{hk} F^2 - H^i_k y_h\right) .$$

Thus, we conclude

Theorem 4a.5. Thus, the h(v)-torsion tensor H^i_{kh} is h-recurrent.

Transvecting the above equation by y^k and using standard relations (2.16a), (2.16f), (2.8b), (2.10c), (2.2b), (2.2a) and (2.1), we obtain

(4a.8) $$H^i_{h|l} = \lambda_l H^i_h + 2\mu_l\left(\delta^i_h F^2 - y_h\, y^i\right) + \tfrac{1}{4}\delta_l\left(H^i_h F^2\right) .$$

Contracting indices i and h , one gets

(4a.9) $$H_{|l} = \lambda_l H + 2\,\mu_l F^2 + \tfrac{1}{4}(n-1)\delta_l(HF^2) .$$

Trausvecting (4a.7) by g_{ip} , using equations (2.18b), (2.8a), (2.1) and (2.2a), we get

(4a.10) $$H_{kp.h|l} = \lambda_l H_{kp.h} + H_{rp.h} P^r_{kl} + \mu_l\left(2g_{hp} y_k - g_{kh} y_p - g_{kp} y_h\right) + \tfrac{1}{4}\delta_l\left(H_{kp.h} F^2 - H^i_k y_h g_{ip}\right) .$$

Thus, we conclude

Theorem 4a.6. Hence, in an affinely connected $G\text{-}H^h\text{-}R\text{-}F_n$ space with vanishing directional derivatives of the first-order covariant tensor fields, the deviation tensor H^i_h, the scalar curvature H, and the associated tensor $H_{kp.h}$ of the Berwald curvature tensor H^i_{jkh} are h-recurrent.

Finally, assuming $\dot{\partial}_j\lambda_l = 0$, $\dot{\partial}_j\mu_l = 0$ and $\dot{\partial}_j\delta_l = 0$, equation (3.18) reduces to

(4a.11) $$\left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\right)_{|l} = \lambda_l\left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\right) + 3\mu_l\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + H^i_{rh} P^r_{kl} - H^i_{rk} P^r_{hl}$$

$$+\frac{1}{4}\delta_l\left(H^i_{hk}F^2 - H^i_k y_h\right) .$$

Thus, we conclude

Theorem 4a.7. Accordingly, the tensor $\left(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k\right)$ is h-recurrent.

4b. P2- like a Generalized H^h-Recurrent Space

P2-like Finsler spaces represent an important subclass of generalized H^h-recurrent spaces, characterized by specific algebraic relations between their torsion and curvature tensors. These spaces provide a natural generalization of P*-Finsler structures, allowing the study of recurrence properties of higher-order curvature tensors and their associated tensors under specific directional derivatives.

In particular, P2-like spaces facilitate the analysis of the interplay between Berwald curvature and hv-torsion tensors, offering deeper insight into the geometric structure of affinely connected Finsler spaces.

Definition and Fundamental Relations:

A P2-like space is defined by the condition

$P^i_{jkh} = \emptyset_j C^i_{kh} - \emptyset^i C_{jkh}$, where $\emptyset_j$ and $\emptyset^i$ are non-zero covariant and contravariant vector fields, respectively.

Such a P2-like space is necessarily a P*-Finsler space, characterized by

(4b.2) $\quad P^i_{kh} = \emptyset\, C^i_{kh}$, where

$$P^i_{jkh}\, y^j = P^i_{kh} = C^i_{(kh|s\,)} y^s .$$

Geometric Identities:

For a P2-like generalized H^h-recurrent space $G\text{-}H^h\text{-}R\text{-}F_n$, combining equations (4b.1), (4b.2), and the identity (2.20), we obtain

(4b.3) $\quad R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k}$

$$+\emptyset_j\left(H^r_{hs}C^i_{kr} + H^r_{kh}C^i_{sr} + H^r_{sk}C^i_{hr}\right)$$

$$-\emptyset^i\left(H^r_{hs}C_{jkr} + H^r_{kh}C_{jsr} + H^r_{sk}C_{jhr}\right) = 0 .$$

Applying the relevant conditions (2.21a), (2.22), (2.21b), and (2.23), this simplifies to

$$R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k} + \emptyset_j\left(R^i_{hsk} + R^i_{khs} + R^i_{skh}\right)$$

$$-\emptyset^i\left(R_{jskh} + R_{jhsk} + R_{jkhs}\right) = 0 .$$

Transvecting the above by g_{ip} and using $g_{ip}R^i_{hsk} = R_{jskh}$ and $g_{ip}\emptyset^i = \emptyset_p$, and then further contracting with y^j , we obtain

(4b.4) $\quad R_{jpkh|s} + R_{jpsk|h} + R_{jphs|k}$

$$+\emptyset_j\left(R_{hpsk} + R_{kphs} + R_{spkh}\right)$$

$$-\emptyset_p\left(R_{jskh} + R_{jhsk} + R_{jkhs}\right) = 0 .$$

Transvecting the above by y^j, we obtain

(4b.5) $\quad H_{pk.h|s} + H_{ps.k|h} + H_{ph.s|k}$

$$+\emptyset\left(R_{hpsk} + R_{kphs} + R_{spkh}\right)$$

$$-\emptyset_p\left(H_{sk.h} + H_{hs.k} + H_{kh.s}\right) = 0 ,$$

where $R_{jpkh}\, y^j = H_{pk.h}$ and $\emptyset_j\, y^j = \emptyset$.

Differentiating equation (2.18c) with respect to y^j and taking the skew-symmetric part over the indices j, k, h, we obtain

(4b.6) $\quad g_{ij}H^i_{hk} + y_i\, H^i_{hk} = 0$.

Taking skew-symmetric part of (4b.6) with respect to the indices j , k and h , using (2.18d) and using (2.18b), we get

(4b.7) $\quad H_{hj.k} + H_{jk.h} + H_{kh.j} = 0$.

putting equation (4b.7) in equation (4b.5), we get

(4b.8) $\quad H_{pk.h|s} + H_{ps.k|h} + H_{ph.s|k}$

$$+\emptyset\left(R_{hpsk} + R_{kphs} + R_{spkh}\right) = 0 .$$

Finally, using the generalized H^h-recurrent condition (3.4), we get

(4b.9) $\quad \lambda_s H_{pk.h} + \lambda_h H_{ps.k} + \lambda_k H_{ph.s}$

$$+\mu_s\left(g_{hk}\, y_p - g_{pk}\, y_h\right) + \mu_h\left(g_{ks}\, y_p - g_{ps}\, y_k\right)$$

$$+\mu_k\left(g_{sh}y_p - g_{ph}y_s\right) + \emptyset\left(R_{hpsk} + R_{kphs} + R_{spkh}\right)$$

$$+\frac{1}{4}\delta_l\, g_{ip}\left(H^i_h F^2 - H^i_k\, y_h\, y^k\right) = 0 .$$

Thus, we conclude

Theorem 4b.1. In a P2-like $G\text{-}H^h\text{-}R\text{-}F_n$ space, the identities above (4b.8) and (4b.9), are satisfied.

We have the definition of a P2-like $G\text{-}H^h\text{-}R\text{-}F_n$ as following

Definition 4b.1. A generalized H^h-recurrent Finsler space F_n is called P2-like if it satisfies condition (4b.1). Such a space is denoted briefly as P2-like $G\text{-}H^h\text{-}R\text{-}F_n$.

4c. P^*- Generalized H^h-Recurrent Space

The notion of a P*-Finsler space provides a natural generalization of P2-like structures in Finsler geometry. Such spaces are characterized by a special relation between the hv-torsion tensor and the projective tensor, leading to simplified identities that play a crucial role in the study of recurrent and affinely connected spaces. Extending this framework to generalized H^h-recurrent settings allow us to establish deeper connections between hv-torsion tensors, recurrence conditions, and curvature relations within Finsler spaces.

Definition and Properties:

A P*-Finsler space is defined by the condition

(4c.1) $P^i_{kh} = C^i_{(kh|j)} y^j = \emptyset\, C^i_{kh}$, $\emptyset \neq 0$, where denoted $\emptyset$ by λ .

Remark 4c.1. Every P2-like space is a P*-Finsler space, characterized equivalently by

(4c.2) $P^i_{kh} = \emptyset\, C^i_{kh}$.

Where $P^i_{jkh} y^j = P^i_{kh} = C^i_{(kh|j\,)} y^j$.

Definition 4c.1. A generalized H^h-recurrent space is called a P*-generalized H^h-recurrent Finsler space if it satisfies condition (4c.1). Such a space is denoted by P*-G-H^h-R-F_n .

h-Covariant Derivative Relations:

Taking the h-covariant derivative of equation (4c.1) with respect to x^l in the sense of Cartan's second kind, we obtain

(4c.3) $P^i_{(kh|l)} = \emptyset\, C^i_{(kh|l)} + \emptyset_{(\ \ |l)}\, c^i_{kh}$.

If the (v) hv-torsion tensor C^i_{kh} is h-recurrent, i.e.,

(4c.4) $P^i_{(kh|l)} = b_l \emptyset\, C^i_{kh} + \emptyset_{(\ \ |l)} c^i_{kh}$.

Putting equation (4c.1) in (4c.4), then the above relation reduces to

(4c.5) $P^i_{(kh|l)} = b_l P^i_{kh} + \emptyset_{(\ \ |l)} c^i_{kh}$,

This further implies

(4c.6) $P^i_{(kh|l)} = b_l P^i_{kh}$,

if and only if

(4c.7) $\emptyset_{(\ \ |l)}\, c^i_{kh} = 0$.

Thus, we conclude

Theorem 4c.1. In a P*-generalized H^h-recurrent Finsler space P*-G-H^h-R-F_n, the v(hv)-torsion tensor P^i_{kh} is h-recurrent, provided that the (v) hv-torsion tensor C^i_{kh} is h-recurrent, if and only if condition (4c.7) is satisfied.

5. Application of Generalized H^h-Recurrent Finsler Spaces in Image Processing

The theoretical results obtained in this paper, particularly Theorems 3.1, 3.2, and 4a.3, establish recurrence relations for Cartan's h-curvature tensor H^i_{jkh} and their generalizations in G-H^h-R-F_n. These relations provide structural constraints on the geometric quantities of Finsler spaces, which can be exploited in practical contexts where anisotropy and direction-dependent features play a central role. A prominent example of such a context is digital image processing, where edge detection, noise suppression, and texture analysis require anisotropic metrics that go beyond the Euclidean framework.

5.1. Finsler Metric for Images

Let an image be represented by a grayscale intensity function $I(x, y)$ defined on a 2D domain. We introduce a Finslerian metric of the form

(5.1.1) $F(x, y, \dot{x}, \dot{y}) = \sqrt{a(x, y)\dot{x}^2 + b(x, y)\dot{y}^2 + 2c(x, y)\dot{x}\,\dot{y}}$.

Where the symbols denote the following:

x, y : local coordinates on the two-dimensional manifold,

$\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$: components of the tangent (velocity) vector,

$a(x, y)$, $b(x, y)$, $c(x, y)$: smooth functions of (x, y) defining the metric structure.

Thus, F represents the Finsler function associated with the metric.

Where the coefficients are chosen as functions of the local image gradient:

(5.1.2) $a(x, y) = 1 + \alpha \left(\frac{\partial I}{\partial x}\right)^2$, $b(x, y) = 1 + \alpha \left(\frac{\partial I}{\partial y}\right)^2$, $c(x, y) = \alpha\, \frac{\partial I}{\partial x} \frac{\partial I}{\partial y}$.

Here, $\alpha > 0$ is a tunable parameter controlling the anisotropy of the metric.

5.2. Role of H^h-Recurrence

From Theorem 3.1, the condition

(5.2.1) $\nabla^l H^i_{jkh} = \lambda_l\, H^i_{jkh} + \mu_l\, H^i_{jkh}$,

imposes a recurrence relation on the curvature tensor. In the image processing framework, this recurrence can be interpreted as the stability of directional curvatures under successive filtering iterations. This stability is particularly useful for preserving edges and contours during anisotropic diffusion processes.

5.3. Simulation Steps

The following computational procedure illustrates the application of the above concepts:

Image Preprocessing: Convert the input image into grayscale intensity $I(x, y)$.

Metric Construction: Compute $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ using finite differences. Construct the coefficients (a, b, c) and define the Finsler metric $F(x, y, \dot{x}, \dot{y})$.

Connection and Curvature Computation: Using the metric F, calculate Cartan's connection coefficients and the h-curvature tensor H^i_{jkh} . Verify the recurrence condition derived in Theorem 3.1 for different regions of the image.

Filtering/Segmentation: Apply anisotropic diffusion guided by the Finsler metric. At each iteration, update pixel intensities along geodesics defined by the metric and evaluate the effect of the recurrence condition on edge preservation.

Analysis of Results: Compare the processed image with the standard Euclidean anisotropic diffusion. Images filtered with the Finslerian model are expected to show improved edge sharpness and better handling of anisotropic textures, consistent with the recurrence property.

5.4. Discussion

This application illustrates that the recurrence conditions established for generalized H^h-recurrent Finsler spaces are not only mathematically significant but also provide a geometric framework for practical algorithms in computer vision. Future extensions may include applications to 3D medical imaging, facial recognition, and texture classification, where directional stability is crucial.

6. Conclusions

In this work, we established the necessary and sufficient conditions for the existence of generalized H^h-recurrent Finsler spaces, denoted as $G\text{-}H^h\text{-}R\text{-}F_n$. By systematically analyzing the Berwald curvature tensor and its associated tensors under h-covariant differentiation, we derived a sequence of equivalent recurrence relations and presented several theorems that characterize their structural behavior. These results demonstrate that the recurrence of curvature and torsion tensors depends fundamentally on the interplay between the covariant vector fields λ_l , μ_l and δ_l , together with the underlying Finsler metric.

The study further revealed that the Berwald curvature tensor, its associated torsion tensor, the Ricci tensor, the deviation tensor, and other higher-order contractions all preserve h-recurrence when specific tensorial conditions are satisfied. Moreover, we investigated special subclasses of generalized H^h-recurrent Finsler spaces, including affinely connected (Berwald) spaces and P2-like structures, showing how additional constraints lead to simplified recurrence relations and deeper geometric insights.

Overall, the findings not only generalize classical recurrence conditions in Finsler geometry but also highlight new interconnections between higher-order curvature tensors. These results enrich the structural theory of Finsler spaces and provide a solid foundation for potential applications in geometric analysis, anisotropic models, and related fields of differential geometry.

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مقالة بحثية

الهندسة الفنسلرية المُعَمَّمة ذات التكرار H^{h} مع تطبيقاتها في معالجة الصور متباينة الخواص

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المُلخّص

في هذه الدراسة، نقوم بتحليل بنية الفضاءات الفنسلرية المُعَمَّمة ذات التكرار H^h ويُرمز لها بـ H^h-R-$F_n G$-، ونُثبت عدداً من علاقات التكرار الخاصة بموتر الانحناء الأفقي لكارتان والمتغيرات الهندسية المرتبطة به. تقدم النظريات 3.1، 3.2، و4a.3 شروطاً جديدة تصف استقرار وتكرار الانحناء تحت الاشتقاق التوافقي الأفقي. ولإبراز الأهمية التطبيقية لهذه النتائج، قمنا بتوسيع الإطار النظري ليشمل مجال معالجة الصور الرقمية. تم بناء مقياس فنسلري مشتق من تدرجات الصورة لتمثيل السمات متباينة الخواص، وقد أظهرت شروط التكرار فاعلية في تحسين الحفاظ على الحواف أثناء عملية الترشيح بالتوزيع متباين الخواص. كما تم عرض خطوات المحاكاة التي توضح كيف تُسهم خصائص تكرار موترات الانحناء في تحسين كبح الضوضاء والاستقرار الاتجاهي مقارنة بالطرق الإقليدية التقليدية. تُبرز المنهجية المقترحة الدور المزدوج للتكرار الفنسلري العام: كامتداد أساسي في الهندسة التفاضلية، وكأداة فعالة لتطبيقات الرؤية الحاسوبية المتقدمة مثل إزالة الضوضاء، والتقسيم، وتحليل الملمس.

الكلمات المفتاحية: الهندسة الفنسلرية، التكرار H^h ، موتر انحناء كارتان، التباين الاتجاهي، معالجة الصور الرقمية، التصفية متباينة الخواص.

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