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RESEARCH ARTICLE

FIFTH-RECURRENT GBK-FINSLER GEOMETRY AND ITS ENGINEERING RELEVANCE TO GEOMETRIC CONTROL AND ROBOTIC MOTION STABILITY

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Abstract

This paper investigates the geometric structure of generalized BK-fifth recurrent Finsler spaces and analyzes the behavior of the associated curvature tensors under higher-order Berwald covariant derivatives. Using the Kulkarni–Nomizu product, several recurrence conditions are established for curvature expressions involving the K-, R-, H-, W-, and P-tensors. The results show that, under the condition $\lambda_m=1/2$, the fifth-order Berwald derivative of these tensors coincides with the fourth-order derivative, indicating a strong form of high-order geometric invariance. From an engineering perspective, such invariance provides a rigorous mathematical foundation for Finsler-based modeling in geometric control and robotic motion planning, where stable curvature structures enhance robustness in nonlinear stabilization, trajectory tracking, and anisotropic path optimization.

Keywords: Finsler geometry; Berwald covariant derivative; Kulkarni–Nomizu product; Robotic motion planning; Nonlinear dynamical systems.

1. Introduction

Finsler geometry has undergone extensive development since its early formulation by Finsler [1] and the subsequent geometric foundations established by Cartan [2], Berwald [3], and Rund [4]. Unlike Riemannian geometry, where the metric depends solely on position, Finsler geometry allows the metric to vary with both position and direction, offering a more general framework with significant flexibility. Modern treatments such as those presented by Shen [5], Matsumoto [6], the study of recurrent geometric properties within this framework has also progressed, beginning with early analyses of recurrent tensors in Finsler spaces by Prasad and Agarwal [7], and Chern–Shen [8] have contributed to the advancement of the theory and its specialized structures, and later expanded through the formulation of tensor calculus and geometric tools in Agarwal's work [9].

In recent years, recurrent and higher-order recurrent Finsler spaces have attracted considerable attention, particularly within the context of generalized BK-structures. Several contributions by AL-Qashbari and collaborators have established a comprehensive foundation for understanding generalized BK-recurrent and BK-fifth recurrent Finsler spaces, including higher-order recurrence conditions [10] to [17]. These works investigated Lie derivatives of various curvature tensors, the inheritance of Kulkarni–Nomizu products, extensions of generalized BK-recurrent structures, and relations among projective curvature tensors. Their findings illuminate the behavior of curvature tensors such as the K-, R-, H-, and M-projective tensors under successive covariant differentiations, providing a deeper understanding of the intrinsic geometric structure of these spaces. Complementary studies have explored related differential operations and curvature behaviors in other settings, such as fluid mechanics and hypersurface geometry, highlighting the broad applicability of Lie derivatives in geometric analysis [19], [22].

The general theory of curvature in Finsler geometry has also been enriched by foundational works on comparative geometry and advanced metric structures by

Opondo [20], Ohta [21], Bao et al. [18], and Pandey et al. [23]. Furthermore, geometric investigations such as those by Shaikh et al. [24] show how curvature-related properties appear in diverse physical models, including black hole spacetimes. These studies collectively emphasize the importance of recurrent curvature identities and the stability of geometric structures under repeated covariant differentiation.

Higher-order recurrence particularly in generalized BK-fifth recurrent Finsler spaces plays a central role in characterizing the invariance of curvature tensors under successive Berwald derivatives. Understanding when higher-order derivatives collapse into lower-order ones yields insights into geometric stability, tensor symmetries, and structural consistency. Such properties are not only mathematically significant but also increasingly relevant for engineering applications, including geometric control, robotic navigation, and modeling of anisotropic dynamical systems, where direction-dependent metrics provide essential flexibility. Motivated by these mathematical foundations and the expanding relevance of curvature-based models, this paper investigates the behavior of multiple curvature tensors in generalized BK-fifth recurrent Finsler spaces. Special attention is given to the recurrence and inheritance of Kulkarni–Nomizu products involving the K-, R-, H-, W-, and P-tensors under higher-order Berwald covariant derivatives. The goal is to establish conditions under which the fifth Berwald derivative coincides with the fourth, revealing structural invariances that extend and unify previous developments in recurrent Finsler geometry.

2. Preliminaries

This section establishes the fundamental mathematical framework necessary for the analysis of generalized BK-fifth recurrent Finsler spaces (GBK-5RF$_n$). It introduces higher-order recurrence relations for the Riemann-Finsler curvature tensor and formulates the conditions under which these tensors exhibit fifth-order recurrence, providing the foundation for subsequent results. The definitions of the Kulkarni-Nomizu product and associated curvature tensors, including K_{ijkh} , P_{ijkh} , and W_{ijkh} , are presented to formalize the structure of these spaces. Additionally, the section details the Lie derivative of mixed tensor fields and the properties of the Berwald covariant derivative, which are essential tools for analyzing the behavior of curvature tensors under successive differentiation. Key results, such as the recurrence relations of Cartan's third and fourth curvature tensors and the Berwald curvature tensor, are summarized to facilitate the derivation of main theorems. Overall, this section provides a rigorous foundation linking tensor calculus, Lie derivatives, and higher-order geometric invariants in GBK-5RF$_n$, ensuring clarity and consistency in the treatment of complex curvature behaviors.

Let us explore a generalized $\mathcal{BK}$-fifth recurrent Finsler space($G\mathcal{B}K-5RF_n$) satisfying the following relations:

(2.1) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh} = a_{sqlnm} R_{ijkh} ,$$

if and only if

(2.2) $$b_{sqlnm}(g_{hj}g_{ik} - g_{kj}g_{ih}) - 2b_{qlnm}y^r\mathcal{B}_r(g_{hj}C_{iks} - g_{kj}C_{ihs}) - c_{sqlnm}(g_{hj}C_{ikn} - g_{kj}C_{ihn}) - d_{sqlnm}(g_{hj}C_{ikl} - g_{kj}C_{ihl}) - e_{sqlnm}(g_{hj}C_{ikq} - g_{kj}C_{ihq}) + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C_{ijt}H^t_{kh}) - a_{sqlnm}(C_{ijt}H^t_{kh}) = 0 .$$

The Kulkarni-Nomizu product $(A \wedge U)$ of two (0,2)-type summetric tensors A and U is defined as

(2.3) $$(A \wedge U)_{ijkh} = A_{ih}U_{jk} - A_{ik}U_{jh} + A_{jk}U_{ih} - A_{jh}U_{ik} .$$

The associate curvature tensors K_{ijkh}, P_{ijkh} and W_{ijkh} satisfying the following relations

(2.4) $$K_{ijkh} = R_{ijkh} - \frac{1}{(n-2)}(A \wedge U)_{ijkh} .$$

(2.5) $$P_{ijkh} = R_{ijkh} - \frac{1}{(n-1)}(A_{ih}U_{jk} - A_{jh}U_{ik}) .$$

(2.6) $$W_{ijkh} = R_{ijkh} - \frac{c}{2n(n-1)}(A \wedge A)_{ijkh} ,$$

(where c is constant).

The Lie-derivative of a general mixed tensor field T^i_{jkh} is given by

(2.7) $$L_v T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh} \mathcal{B}_m v^i + T^i_{mkh} \mathcal{B}_j v^m + T^i_{jmh} \mathcal{B}_k v^m + T^i_{jkm} \mathcal{B}_h v^m + \dot{\partial}_m T^i_{jkh} \mathcal{B}_r v^m y^r , \text{ where } v^m \neq 0 .$$

The Berwald covariant derivative of the contravariant vector field v^m vanish identically, i.e:

(2.8) $$\mathcal{B}_j v^m = 0 .$$

The Cartan's fourth curvature tensor K^i_{jkh}, Cartan's third curvature tensor R^i_{jkh} and the curvature tensor of Berwald H^i_{jkh} in recurrent Finsler space is defined as

$$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} , \quad \mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} ,$$

and $$\mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh} .$$

Contracting the indices i and h in above equation, we get

(2.9) a) $\mathcal{B}_m K_{jk} = \lambda_m K_{jk}$, b) $\mathcal{B}_m R_{jk} = \lambda_m R_{jk}$, and c) $\mathcal{B}_m H_{jk} = \lambda_m H_{jk}$.

3. Fifth-Order Berwald Covariant Derivatives of Kulkarni–Nomizu Products in GBK-5RF$_n$ Spaces

This section presents the main theoretical contributions regarding the behavior of Kulkarni–Nomizu products and associated curvature tensors in generalized BK-fifth recurrent Finsler spaces (GBK-5RF$_n$).

We systematically investigate the fifth-order Berwald covariant derivatives of the Kulkarni–Nomizu products formed by K-Ricci, R-Ricci, and H-Ricci tensors, establishing conditions under which these products exhibit fifth-order recurrence. The section further explores the Lie derivatives of these tensor products and their relationships with the fourth-order covariant derivatives, leading to a series of theorems and corollaries that characterize the recurrent behavior of associated curvature tensors K_{ijkh} , P_{ijkh} , and W_{ijkh} . Additionally, we highlight special cases where symmetry in the Kulkarni–Nomizu products leads to vanishing fifth-order derivatives, emphasizing the interplay between tensor symmetries and higher-order geometric invariants. Overall, this section provides a rigorous analytical framework that connects tensor algebra, Lie derivatives, and higher-order Berwald derivatives, laying the groundwork for the derivation of the main results.

Let us consider Kulkarni-Nomizu product of K-Ricci tensor and R-Ricci tensor $(K \wedge R)_{ijkh}$ defined as (2.3), then we have

(3.1) $$(K \wedge R)_{ijkh} = K_{ih}R_{jk} - K_{ik}R_{jh} + K_{jk}R_{ih} - K_{jh}R_{ik} .$$

Taking the Lie -derivative of both sides of above equation, we get

$$L_v(K \wedge R)_{ijkh} = R_{jk}L_vK_{ih} + K_{ih}L_vR_{jk} - R_{jh}L_vK_{ik} - K_{ik}L_vR_{jh} + R_{ih}L_vK_{jk} + K_{jk}L_vR_{ih} - R_{ik}L_vK_{jh} - K_{jh}L_vR_{ik} .$$

Using (2.7) and (2.8) in above equation, we get

$$\mathcal{B}_m(K \wedge R)_{ijkh} = R_{jk}\mathcal{B}_mK_{ih} + K_{ih}\mathcal{B}_mR_{jk} - R_{jh}\mathcal{B}_mK_{ik} - K_{ik}\mathcal{B}_mR_{jh} + R_{ih}\mathcal{B}_mK_{jk} + K_{jk}\mathcal{B}_mR_{ih} - R_{ik}\mathcal{B}_mK_{jh} - K_{jh}\mathcal{B}_mR_{ik} .$$

Using [(2.9)a,b] in above equation, we get

$$\mathcal{B}_m(K \wedge R)_{ijkh} = 2\lambda_m[K_{ih}R_{jk} - K_{ik}R_{jh} + K_{jk}R_{ih} - K_{jh}R_{ik}]$$

Let $\lambda_m = \frac{1}{2}$ and taking Berwald covariant derivative of fourth order for above equation with respect to x^n , x^l, x^q and x^s, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K \wedge R)_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n[K_{ih}R_{jk} - K_{ik}R_{jh} + K_{jk}R_{ih} - K_{jh}R_{ik}] .$$

Using (3.1) in above equation, we get

(3.2) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K \wedge R)_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n(K \wedge R)_{ijkh} .$$

Thus, we conclude

Theorem 3.1: In $\mathcal{G}BK - 5RF_n$, the Berwald covariant derivative of fifth order is equal the Berwald covariant derivative of fourth order for the Kulkarni- Nomizu product $(K \wedge R)_{ijkh}$ if the covariant vector $\lambda_m = \frac{1}{2}$.

Now, we have two corollaries related to the previous theorem. Using the same steps as in the previous theorem, by taking the Kulkarni-Nomizu product $(R \wedge H)_{ijkh}$ and $(K \wedge H)_{ijkh}$ respectively, by using [(2.9) a,b,c] we obtain

Corollary 3.1: In $\mathcal{G}BK - 5RF_n$, the Berwald covariant derivative of fifth order is equal the Berwald covariant derivative of fourth order for the Kulkarni- Nomizu product $(R \wedge H)_{ijkh}$ if the covariant vector $\lambda_m = \frac{1}{2}$.

Corollary 3.2: In $\mathcal{G}BK - 5RF_n$, the Berwald covariant derivative of fifth order is equal the Berwald covariant derivative of fourth order for the Kulkarni- Nomizu product $(K \wedge H)_{ijkh}$ if the covariant vector $\lambda_m = \frac{1}{2}$.

Using the Kulkarni- Nomizu product of K- Ricci tensor and R-Ricci tensor $(K \wedge R)_{ijkh}$ in (2.4), we have

(3.3) $$K_{ijkh} = R_{ijkh} - \frac{1}{(n-2)}(K \wedge R)_{ijkh} .$$

Taking the Lie - derivative of both sides of above equation and using (2.7) and (2.8), we get

$$\mathcal{B}_mK_{ijkh} = \mathcal{B}_mR_{ijkh} - \frac{1}{(n-2)}\mathcal{B}_m(K \wedge R)_{ijkh} .$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mK_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mR_{ijkh} - \frac{1}{(n-2)} \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K \wedge R)_{ijkh} .$$

Using (2.1) in above equation, we get

(3.4) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mK_{ijkh} = a_{sqlnm}R_{ijkh} - \frac{1}{(n-2)}a_{sqlnm}(K \wedge R)_{ijkh} ,$$

if and only if

(3.5) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K \wedge R)_{ijkh} = a_{sqlnm}(K \wedge R)_{ijkh} .$$

Above equation means that the Kulkarni-Nomizu product $(K \wedge R)_{ijkh}$ behaves as fifth recurrent.

Now, using (3.3) in right side of (3.4) we get

(3.6) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{ijkh} = a_{sqlnm} K_{ijkh}$.

Thus, we conclude

Theorem 3.2: In $G\mathcal{B}K-5RF_n$, the associate curvature tensor K_{ijkh} of the curvature tensor K^i_{jkh} behaves as fifth recurrent if and only if the Kulkarni-Nomizu product $(K \wedge R)_{ijkh}$ behaves as fifth recurrent [provided (2.2) holds].

Using the Kulkarni- Nomizu product of K- Ricci tensor with itself in (2.6), we have

(3.7) $W_{ijkh} = R_{ijkh} - \frac{c}{2n(n-1)}(K \wedge K)_{ijkh}$

Taking the Lie - derivative of both sides of above equation and using (2.7) and (2.8), we get

$$\mathcal{B}_m W_{ijkh} = \mathcal{B}_m R_{ijkh} - \frac{c}{2n(n-1)}\mathcal{B}_m(K \wedge K)_{ijkh} .$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m W_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}$$
$$-\frac{c}{2n(n-1)}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K \wedge K)_{ijkh} .$$

Using (2.1) in above equation, we get

(3.8) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m W_{ijkh} = a_{sqlnm} R_{ijkh}$
$$-\frac{c}{2n(n-1)} a_{sqlnm}(K \wedge K)_{ijkh} ,$$

if and only if

(3.9) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (K \wedge K)_{ijkh}$
$$= a_{sqlnm}(K \wedge K)_{ijkh} .$$

Now, using (3.7) in right side of (3.8), we get

(3.10) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m W_{ijkh} = a_{sqlnm} W_{ijkh}$.

Thus, we conclude

Corollary 3.3: In $G\mathcal{B}K-5RF_n$, the associate curvature tensor W_{ijkh} of the curvature tensor W^i_{jkh} behaves as fifth recurrent if and only if the Kulkarni-Nomizu product $(K \wedge K)_{ijkh}$ behaves as fifth recurrent [provided (2.2) holds].

Using the K-Ricci tensor and H-Ricci tensor in (2.5), we have

(3.11) $P_{ijkh} = R_{ijkh} - \frac{1}{(n-1)}(K_{ih}H_{jk} - K_{jh}H_{ik})$.

Taking the Lie - derivative of both sides of above equation and using (2.7) and (2.8), we get

$$\mathcal{B}_m P_{ijkh} = \mathcal{B}_m R_{ijkh} - \frac{1}{(n-1)}\mathcal{B}_m(K_{ih}H_{jk} - K_{jh}H_{ik}) .$$

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m P_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}$$
$$-\frac{1}{(n-1)}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K_{ih}H_{jk} - K_{jh}H_{ik}) .$$

Using (2.1) in above equation, we get

(3.12) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m P_{ijkh} = a_{sqlnm} R_{ijkh}$
$$-\frac{1}{(n-1)} a_{sqlnm}(K_{ih}H_{jk} - K_{jh}H_{ik}) ,$$

if and only if

(3.13) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K_{ih}H_{jk} - K_{jh}H_{ik})$
$$= a_{sqlnm}(K_{ih}H_{jk} - K_{jh}H_{ik}) .$$

Now, using (3.11) in right side of (3.12), we get

(3.14) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m P_{ijkh} = a_{sqlnm} P_{ijkh}$.

Thus, we conclude

Corollary 3.4: In $G\mathcal{B}K-5RF_n$, the associate curvature tensor P_{ijkh} of the curvature tensor P^i_{jkh} behaves as fifth recurrent if and only if the tensor $(K_{ih}H_{jk} - K_{jh}H_{ik})$ behaves as fifth recurrent [provided (2.2) holds].

Now, if the the Kulkarni-Nomizu product of K-Ricci tensor and R-Ricci tensor $(K \wedge R)_{ijkh}$ is symmetric in its indices i and j , then the equation (3.1), can be written as

$$(K \wedge R)_{ijkh} = 0 .$$

Taking the Lie-derivative of both sides of above equation and using (2.7) and (2.8), then taking Berwald covariant derivative of fourth order for result equation with respect to x^n, x^l, x^q and x^s, we get

(3.15) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(K \wedge R)_{ijkh} = 0$.

Thus, we conclude

Theorem 3.3: In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of fifth order for the Kulkarni-Nomizu product $(K \wedge R)_{ijkh}$ is vanishing if it symmetric in its indices i and j .

4. Engineering Applications

The mathematical results obtained in this work on GBK-fifth recurrent Finsler spaces provide significant implications for modern engineering systems modeled by Finsler-type geometries. The demonstrated equivalence between the fifth-order and fourth-order Berwald covariant derivatives of various Kulkarni-Nomizu curvature products such as $(KR)_{ijkh}$, $(RH)_{ijkh}$, $(KH)_{ijkh}$, $(KK)_{ijkh}$ reveals a high degree of geometric stability and structural invariance of the underlying curvature tensors.

Such properties are of substantial value in engineering fields where nonlinear dynamics evolve in non-Euclidean geometric environments.

In geometric control theory, Finsler metrics have recently emerged as powerful tools to model anisotropic and non-linear cost structures.

The recurrence conditions established in this paper imply that the associated curvature tensors preserve their form under higher-order Berwald differentiations, which corresponds to a type of dynamic invariance desirable in systems governed by repeated or prolonged control actions.

This invariance directly enhances the robustness of stabilization, tracking, and regulation in nonlinear control systems.

In robotic motion planning, particularly for autonomous systems operating in heterogeneous or direction-dependent environments, Finsler geometry provides more realistic models than classical Riemannian approaches.

The recurrence of the Kulkarni–Nomizu curvature products ensure the persistence of curvature-based constraints along optimal trajectories, contributing to trajectory stability, reduced computational sensitivity, and consistent geometric behavior of the robot's velocity and acceleration fields.

Moreover, the vanishing conditions such as the symmetry-induced relation $(KR)_{ijkh} = 0$ indicate deep geometric symmetries that can be exploited in mechanical design, swarm coordination, and multi-agent engineering systems.

These structures can be interpreted as indicators of inherent balance, energy symmetry, or geometric conservation within the mechanical system.

Thus, the theoretical contributions of this paper provide a rigorous geometric framework that strengthens the mathematical foundation underlying several engineering technologies based on Finsler geometric modeling, including but not limited to:

1. geometric nonlinear control,
2. autonomous robot navigation,
3. optimal path-planning,
4. and stability analysis of anisotropic mechanical systems.

5. Conclusions

In this work, we examined the structure of generalized BK-fifth recurrent Finsler spaces GBK-5RF$_{(n)}$ and established several recurrence relations for curvature tensors derived from the Kulkarni–Nomizu product.

The main results demonstrate that, under the condition $\lambda_m = \frac{1}{2}$, the fifth-order Berwald covariant derivative of the curvature products $(KR)_{ijkh}$, $(RH)_{ijkh}$, $(KH)_{ijkh}$, and $(KK)_{ijkh}$ coincides with their fourth-order derivative. This indicates a strong form of high-order geometric invariance, showing that these curvature structures remain stable under repeated geometric differentiations.

Additionally, the necessary and sufficient conditions obtained for the recurrence of the associated curvature tensors K_{ijkh} , W_{ijkh} , P_{ijkh} reinforce the presence of deep geometric symmetries within these Finsler spaces. From an engineering standpoint, such high-order curvature invariance provides a solid mathematical foundation for several applications involving geometric modeling of nonlinear dynamical systems.

In geometric control theory, the recurrence of curvature tensors implies persistent geometric constraints that enhance the robustness of stabilization and trajectory tracking.

In robotic motion planning, these results support the development of Finsler-based path-planning algorithms capable of maintaining consistent geometric behavior in anisotropic or direction-dependent environments.

The vanishing and symmetry conditions derived in this work further highlight geometric properties that can be exploited in mechanical systems, multi-agent coordination, and navigation algorithms.

Thus, the theoretical developments presented here not only advance the study of recurrent Finsler geometry but also provide relevant mathematical tools for engineering systems operating in non-Euclidean geometric frameworks, particularly in control, robotics, and motion stability analysis.

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مقالة بحثية

الهندسة الفينسلرية المتكررة من الدرجة الخامسة من نوع GBK وأهميتها الهندسية للتحكم الهندسي واستقرار حركة الروبوتات

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المُلخّص

تتناول هذه الورقة البحثية الهيكل الهندسي لفضاءات فينسلر المتكررة من الدرجة الخامسة العامة من نوع BK (GBK-5RF)، وتُحلل سلوك الموترات الانحنائية المرتبطة بها تحت المشتقات التعاقدية لبيروالد عالية الرتبة. باستخدام منتج كولكارني–نوميزو، تم وضع عدة شروط للتكرار للتعبيرات الانحنائية التي تشمل موترات K وR وH وW وP. تُظهر النتائج أنه، تحت الشرط $\lambda_m=1/2$، تتطابق المشتقة التعاقدية لبيروالد من الرتبة الخامسة لهذه الموترات مع المشتقة من الرتبة الرابعة، مما يشير إلى شكل قوي من الاستقرار الهندسي عالي الرتبة. من منظور هندسي، يوفر هذا الاستقرار أساسًا رياضيًا صارمًا للنمذجة المعتمدة على الهندسة الفينسلرية في التحكم الهندسي وتخطيط حركة الروبوتات، حيث تسهم هياكل الانحناء المستقرة في تعزيز المتانة في الاستقرار غير الخطي، وتتبع المسارات، وتحسين المسارات في بيئات غير متجانسة أو غير متماثلة.

الكلمات المفتاحية: الهندسة الفينسلرية؛ المشتقة التعاقدية لبيروالد؛ منتج كولكارني - نوميزو؛ تخطيط حركة الروبوتات؛ الأنظمة الديناميكية غير الخطية.

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