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RESEARCH ARTICLE

THE GENERALIZED Q-BESSEL MATRIX FUNCTION OF TWO VARIABLES

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Abstract

The Bessel function is probably the best known special function, within pure and applied mathematics. In this paper, we introduce the generalized q-analogue Bessel matrix function of two variables. Some properties of this function, such as generating function, q-difference equation, and recurrence relations are obtained.

Keywords: Q-analogue Bessel matrix function of two variables, Generating function, The q-difference equation and the recurrence relations.

1. Introduction.

The theory of special functions performs an essential role in the formalism of mathematical physics. The Bessel functions [10] are one of the most important special functions and have applications in number theory, lie theory and theoretical astronomy to some problems of engineering and physics.

The Bessel's function of first kind and r order is defined by [10]

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+r}, \quad (1.1)$$

and the q-analogue Bessel function of one variable is defined by [3, 4]

$$J_r(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q![r+n]_q!} \left(\frac{x}{2}\right)^{2n+r}, \quad (1.2)$$

$$\text{where} \quad \lim_{q\to 1} J_r\big((1-q)x;q\big) = J_r(x). \quad (1.3)$$

Mahmoud [9] presented the following generalized q-Bessel function:

$$J_n(x,a;q) = \frac{(x/2)^n}{(q;q)_n} \sum_{k=0}^{\infty} (-1)^{k(a+1)} \frac{q^{\frac{ak}{2}(k+n)}}{(q^{n+1};q)_n} \frac{(x^2/4)^k}{(q;q)_k}, \quad (1.4)$$

which converges absolutely for all x when $a \in Z^+$ and for $|x| < 2$ if $a = 0$.

The Bessel matrix functions $J_A(z)$ of the first kind of order A is defined as follows [11, 12]:

$$J_A(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma^{-1}(A+(n+1)I) \left(\frac{z}{2}\right)^{A+2nI}, \quad (1.5)$$

where A is a matrix such that $A \in C^{N\times N}$ satisfying the condition μ is not a negative integer for all $\mu \in \sigma(A)$, where $\sigma(A)$ is the set of all eigenvalue of A.

The two variable Bessel's functions are defined by the following series representations [1]:

$$J_{r,s}(x,y) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}}{n!m!\Gamma(r+m+1)\Gamma(s+n+1)} \left(\frac{x}{2}\right)^{2m+r} \left(\frac{yp(x)}{2}\right)^{2n+s}, \quad (1.6)$$

and

$$J_{r,s}(x,y) = \frac{\left(\frac{x}{2}\right)^r \left(\frac{yp(x)}{2}\right)^s}{\Gamma(r+1)\Gamma(s+1)} \, {}_0F_1\left(-;r+1;-\frac{x^2}{4}\right) {}_0F_1\left(-;s+1;-\frac{y^2p^2(x)}{4}\right), \quad (1.7)$$

respectively, where r and s both integers.

Also, Tenguria and Sharma [13] introduced and studied the advanced q-Bessel function of two variables defined by

$$J_{r,s}(x,y;q) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}}{[m]_q![r+m]_q![n]_q![s+n]_q!} \left(\frac{x}{2}\right)^{2m+r} \left(\frac{yp(x)}{2}\right)^{2n+s}, \quad (1.8)$$

$$\text{where} \quad \lim_{q\to 1} J_{r,s}\big((1-q)x,(1-q)y;q\big) = J_{r,s}(x,y).$$

The q-shifted factorials are defined by [7]

$$(a;q)_n = \begin{cases} 1, & n=0 \\ \prod_{k=0}^{n-1}(1-aq^k), & n \in N' \end{cases} \quad (1.9)$$

$$(a_1,\dots,a_r;q)_k = \prod_{i=1}^{r}(a_i;q)_k \quad ; k = 0,1,2,\dots \quad (1.10)$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \quad (1.11)$$

where $a, a_i's, q \in R$ such that $0 < q < 1$.

Exton [5] presented the whole family of basic q-exponential function as:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} q^{\mu n(n-1)} \frac{z^n}{[n]_q!}, \quad (1.12)$$

where $$[n]_q! = \frac{(q;q)_n}{(1-q)^n}. \quad (1.13)$$

The one parameter family of q-exponential functions

$$E_q^{(\alpha)}(z) = \sum_{n=0}^{\infty} q^{\alpha n^2/2} \frac{z^n}{(q;q)_n},$$

with $\alpha \in R$ has been considered in [6]. Consequently, in the limit when $q \to 1$,

we have $\lim_{q \to 1} E_q^{(\alpha)}((1-q)z) = e^z$.

In Exton's formula, if we replace z by $\frac{x}{1-q}$ and μ by $\frac{a}{2}$, we get

$$E\left(\frac{a}{2}, \frac{x}{1-q}; q\right) = E_q(x, a),$$

where

$$E_q(x, a) = \sum_{n=0}^{\infty} q^{a\binom{n}{2}} \frac{x^n}{(q;q)_n}, \quad (1.14)$$

which satisfies the functional relation [2]

$$E_q(x, a) - E_q(qx, a) = xE_q(q^a x, a).$$

The above q-function can be rewritten by the formula

$$D_q E_q(x, a) = \frac{1}{1-q} E_q(q^a x, a). \quad (1.15)$$

Where the Jackson q-difference operator D_q is defined by [8]

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (1.16)$$

and satisfies the product rule

$$D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x). \quad (1.17)$$

In this paper, we introduce a second form of the generalized q-Bessel matrix function of two variables and study some of its properties.

2. The generalized q-Bessel Matrix function of two variables

We define the generalized q-Bessel matrix function of two variables, denoted by $J_{r,s}(x, y, a, A, B; q)$, by the following generating function:

$$E_q\left[\frac{q^{a/4} x\sqrt{2A}t}{2}, \frac{a}{2}\right] E_q\left[\frac{(-1)^{a+1} q^{a/4} x\sqrt{2A}}{2t}, \frac{a}{2}\right] E_q\left[\frac{q^{a/4} Byp(x)w}{2}, \frac{a}{2}\right]$$

$$\times E_q\left[\frac{(-1)^{a+1} q^{a/4} Byp(x)}{2w}, \frac{a}{2}\right] = \sum_{r,s=-\infty}^{\infty} q^{a/4(r^2+s^2)} J_{r,s}(x, y, a, A, B; q) t^r w^s, \quad (2.1)$$

where $a \in Z^+, x, y \in R,\ t, w \in \mathbb{C},\ t, w \neq 0$

$A, B \in C^{N\times N}, x > 0, p(x) > 0$, satisfying the condition of the matrix in (1.5).

Now, by using the above generating function we will deduce the generalized q-Bessel function of two variables $J_{r,s}(x, y, a, A, B; q)$ in the form of the following theorem:

Theorem 2.1.

Let us assume that $A, B \in C^{N\times N}, x > 0, p(x) > 0,\ 0 < q < 1$ then the following formula for q-Bessel function of two variables $J_{r,s}(x, y, a, A, B; q)$ holds true:

$$J_{r,s}(x, y, a, A, B; q) = \frac{\left(\frac{x\sqrt{2A}}{2}\right)^r \left(\frac{Byp(x)}{2}\right)^s}{(q;q)_r (q;q)_s}$$

$$\times \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)} q^{a/2[m(m+r)+n(n+s)]}}{(q^{1+r};q)_m (q;q)_m (q^{1+s};q)_n (q;q)_n} \left(\frac{x\sqrt{2A}}{2}\right)^{2m} \left(\frac{Byp(x)}{2}\right)^{2n}, \quad (2.2)$$

which converges absolutely for all x and y when $a \in Z^+$ and for $|x| < 2, |y| < 2$ if $a = 0$.

Proof. Let us denote the left hand side of (2.1) by W and by using (1.14), we have

$$W = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{m(a+1)} \frac{q^{a/2[\binom{r}{2}+\binom{m}{2}]+\frac{a}{4}(r+m)}}{(q;q)_r (q;q)_m} \left(\frac{x\sqrt{2A}}{2}\right)^{r+m} t^{r-m}$$

$$\times \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{n(a+1)} \frac{q^{a/2[\binom{s}{2}+\binom{n}{2}]+\frac{a}{4}(s+n)}}{(q;q)_s (q;q)_n} \left(\frac{Byp(x)}{2}\right)^{s+n} w^{s-n}, \quad (2.3)$$

Replace r and s by $r + m$ and $s + n$ respectively in the right hand side of equation (2.3), we get

$$W = \sum_{r=-\infty}^{\infty} \sum_{m=0}^{\infty} (-1)^{m(a+1)} \frac{q^{a/2[\binom{r+m}{2}+\binom{m}{2}]+\frac{a}{4}(r+2m)}}{(q;q)_{r+m} (q;q)_m} \left(\frac{x\sqrt{2A}}{2}\right)^{r+2m} t^r$$

$$\times \sum_{s=-\infty}^{\infty} \sum_{n=0}^{\infty} (-1)^{n(a+1)} \frac{q^{a/2[\binom{s+n}{2}+\binom{n}{2}]+\frac{a}{4}(s+2n)}}{(q;q)_{s+n} (q;q)_n} \left(\frac{Byp(x)}{2}\right)^{s+2n} w^s, \quad (2.4)$$

By using relation (1.11) in (2.4), we obtain

$$W = \sum_{r,s=-\infty}^{\infty} \frac{q^{a/4(r^2+s^2)}}{(q;q)_r (q;q)_s} \left(\frac{x\sqrt{2A}}{2}\right)^r \left(\frac{Byp(x)}{2}\right)^s$$

$$\times \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)} q^{a/2[m(m+r)+n(n+s)]}}{(q^{1+r};q)_m (q;q)_m (q^{1+s};q)_n (q;q)_n} \left(\frac{x\sqrt{2A}}{2}\right)^{2m} \left(\frac{Byp(x)}{2}\right)^{2n} t^r w^s. \quad (2.5)$$

By equating the coefficients of $t^r w^s$ with the right hand side of (2.1), we get the relation (2.2).

Remark 2.1. Putting $a = 0, A = \frac{1}{2}I, B = I$ and $y = 0, A = \frac{1}{2}I, B = I$ in equation (2.1) and in view of equations (1.8) and (1.4), we get

$$J_{r,s}\left(x, y, 0, \frac{1}{2}I, I; q\right) = J_{r,s}(x, y; q), \quad (2.6)$$

and

$$J_{r,s}\left(x, a, 0, \frac{1}{2}I, I; q\right) = J_r(x, a; q), \quad (2.7)$$

respectively.

Corollary 2.1.

The function $J_{r,s}(x, y, a, A, B; q)$ is a q-analogy of each of the Bessel matrix function and the modified Bessel matrix function.

Proof. Using (1.11) and (1.15) in (2.2) and taking the limit as $q \to 1$ and setting $x \to (1-q)x$ and $y \to (1-q)$y, we get

$$\lim_{q\to 1} J_{r,s}\big((1-q)x, (1-q)y, a, A, B; q\big)$$

$$= \lim_{q\to 1} \frac{\left(\frac{(1-q)x\sqrt{2A}}{2}\right)^r \left(\frac{(1-q)Byp(x)}{2}\right)^s}{(q;q)_r (q;q)_s}$$

$$\times \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)} q^{a/2[m(m+r)+n(n+s)]}}{(q^{1+r};q)_m (q;q)_m (q^{1+s};q)_n (q;q)_n} \left(\frac{(1-q)x\sqrt{2A}}{2}\right)^{2m} \left(\frac{(1-q)Byp(x)}{2}\right)^{2n}$$

$$= \left(\frac{x\sqrt{2A}}{2}\right)^r \left(\frac{Byp(x)}{2}\right)^s \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)}}{\Gamma(r+m+1)m!\Gamma(s+n+1)n!} \left(\frac{x\sqrt{2A}}{2}\right)^{2m} \left(\frac{Byp(x)}{2}\right)^{2n}$$

Hence,

$$\lim_{q\to 1} \quad J_{r,s}((1-q)x,(1-q)y,a,A,B;q) = J_{r,s}(x,y,a,A,B) \;, \quad (2.8)$$

corollary 2.2.

If r, s be integer, then $J_{r,s}(x,y,a,A,B;q)$ satisfies the following relations:

$$J_{-r,s}(x,y,a,A,B;q) = (-1)^{r(a+1)} J_{r,s}(x,y,a,A,B;q) \;, \quad (2.9)$$

$$J_{r,-s}(x,y,a,A,B;q) = (-1)^{s(a+1)} J_{r,s}(x,y,a,A,B;q) \;, \quad (2.10)$$

$$J_{-r,-s}(x,y,a,A,B;q) = (-1)^{(r+s)(a+1)} J_{r,s}(x,y,a,A,B;q) \;, \quad (2.11)$$

Proof. Using (2.2), we get

$$J_{-r,s}(x,y,a,A,B;q)$$

$$= \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)} q^{a/2[m(m-r)+n(n+s)]}}{(q;q)_{m-r}(q;q)_m(q;q)_{n+s}(q;q)_n} \left(\frac{x\sqrt{2A}}{2}\right)^{2m-r} \left(\frac{Byp(x)}{2}\right)^{2n+s} , \quad (2.12)$$

replace m by $m+r$ in the r.h.s. of (2.12) , we obtain

$$J_{-r,s}(x,y,a,A,B;q)$$

$$= \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+r+n)(a+1)} q^{a/2[m(m+r)+n(n+s)]}}{(q;q)_{m+r}(q;q)_m(q;q)_{n+s}(q;q)_n} \left(\frac{x\sqrt{2A}}{2}\right)^{2m+r} \left(\frac{Byp(x)}{2}\right)^{2n+s} , \quad (2.13)$$

which on using definition (2.2) gives yields the required relation(2.9).

Similarly,we can prove the relations (2.10) and (2.11).

corollary 2.3.

The function $J_{r,s}(x,y,a,A,B;q)$ satisfies the relations:

$$J_{r,s}(-x,y,a,A,B;q) = (-1)^r J_{r,s}(x,y,a,A,B;q), \quad (2.14)$$

$$J_{r,s}(x,-y,a,A,B;q) = (-1)^s J_{r,s}(x,y,a,A,B;q), \quad (2.15)$$

$$J_{r,s}(-x,-y,a,A,B;q) = (-1)^{r+s} J_{r,s}(x,y,a,A,B;q). \quad (2.16)$$

Proof. Since,

$$J_{r,s}(-x,y,a,A,B;q) = \frac{\left(-\frac{x\sqrt{2A}}{2}\right)^r \left(\frac{Byp(x)}{2}\right)^s}{(q;q)_r(q;q)_s}$$

$$\times \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)} q^{a/2[m(m+r)+n(n+s)]}}{(q^{1+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n} \left(-\frac{x\sqrt{2A}}{2}\right)^{2m} \left(\frac{Byp(x)}{2}\right)^{2n}$$

$$= (-)^r \frac{\left(\frac{x\sqrt{2A}}{2}\right)^r \left(\frac{Byp(x)}{2}\right)^s}{(q;q)_r(q;q)_s} \sum_{m,n=0}^{\infty} \frac{(-1)^{(m+n)(a+1)} q^{a/2[m(m+r)+n(n+s)]}}{(q^{1+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n} \left(\frac{x\sqrt{2A}}{2}\right)^{2m} \left(\frac{Byp(x)}{2}\right)^{2n}$$

which in view of (2.2) yields relation (2.14).

Similarly, the relations (2.15) and (2.16), can be proved.

3. The q-difference equation of the matrix function $J_{r,s}(x,y,a,A,B;q)$

In order to derive the q-difference equation of the function $J_{r,s}(x,y,a,A,B;q)$ applying the operator $D_{y,q}$, on both sides of generating function (2.1) and using relations (1.15) and (1.17), we get

$$E_q\left[\frac{q^{a/4}x\sqrt{2A}t}{2}, \frac{a}{2}\right] E_q\left[\frac{(-1)^{a+1}q^{a/4}x\sqrt{2A}}{2t}, \frac{a}{2}\right]$$

$$\left\{\frac{(-1)^{a+1}q^{a/4}Bp(x)}{2(1-q)w} E_q\left[\frac{q^{a+4/4}Byp(x)w}{2}, \frac{a}{2}\right] E_q\left[\frac{(-1)^{a+1}q^{3a/4}Byp(x)}{2w}, \frac{a}{2}\right] + \frac{q^{a/4}Bp(x)w}{2(1-q)} E_q\left[\frac{q^{3a/4}Byp(x)w}{2}, \frac{a}{2}\right] E_q\left[\frac{(-1)^{a+1}q^{a/4}Byp(x)}{2w}, \frac{a}{2}\right]\right\}$$

$$= \sum_{r,s=-\infty}^{\infty} q^{a/4(r^2+s^2)} D_{y,q} J_{r,s}(x,y,a,A,B;q) t^r w^s \;. \quad (3.1)$$

Using (1.14) in the l.h.s. of the above equation, we obtain

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m(a+1)} \frac{q^{a/2[\binom{r}{2}+\binom{m}{2}]+\frac{a}{4}(r+m)}}{(q;q)_r(q;q)_m} \left(\frac{x\sqrt{2A}}{2}\right)^{r+m} t^{r-m}$$

$$\left\{\frac{(-1)^{a+1}q^{a/4}Bp(x)}{2(1-q)w} \sum_{s=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{n(a+1)} \frac{q^{a/2[\binom{s}{2}+\binom{n}{2}]+\frac{a}{4}(s+n)+s+\frac{a}{2}n}}{(q;q)_s(q;q)_n} \left(\frac{Byp(x)}{2}\right)^{s+n} w^{s-n} + \frac{q^{a/4}Bp(x)w}{2(1-q)} \sum_{s=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{n(a+1)} \frac{q^{a/2[\binom{s}{2}+\binom{n}{2}]+\frac{a}{4}(s+n)+\frac{a}{2}s}}{(q;q)_s(q;q)_n} \left(\frac{Byp(x)}{2}\right)^{s+n} w^{s-n}\right\}$$

$$= \sum_{r,s=-\infty}^{\infty} q^{a/4(r^2+s^2)} D_{y,q} J_{r,s}(x,y,a,A,B;q) t^r w^s \;. \quad (3.2)$$

Replace r and s by $r+m$ and $s+n$ respectively in the l.h.s. of (3.2), we get

$$\sum_{r=-\infty}^{\infty}\sum_{m=0}^{\infty}(-1)^{m(a+1)} \frac{q^{a/2[\binom{r+m}{2}+\binom{m}{2}]+\frac{a}{4}(r+2m)}}{(q;q)_{r+m}(q;q)_m} \left(\frac{x\sqrt{2A}}{2}\right)^{r+2m} t^r$$

$$\left\{\frac{(-1)^{a+1}q^{a/4}Bp(x)}{2(1-q)w} \sum_{s=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{n(a+1)} \frac{q^{a/2[\binom{s+n}{2}+\binom{n}{2}]+\frac{a}{4}(s+2n)+s+n+\frac{a}{2}n}}{(q;q)_{s+n}(q;q)_n} \left(\frac{Byp(x)}{2}\right)^{s+2n} w^s + \frac{q^{a/4}Bp(x)w}{2(1-q)} \sum_{s=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{n(a+1)} \frac{q^{a/2[\binom{s+n}{2}+\binom{n}{2}]+\frac{a}{4}(s+2n)+\frac{a}{2}(s+n)}}{(q;q)_{s+n}(q;q)_n} \left(\frac{Byp(x)}{2}\right)^{s+2n} w^s\right\}$$

$$= \sum_{r,s=-\infty}^{\infty} q^{a/4(r^2+s^2)} D_{y,q} J_{r,s}(x,y,a,A,B;q) t^r w^s. \quad (3.3)$$

which on using definition (2.2) in the l.h.s. gives

$$\sum_{r,s=-\infty}^{\infty} q^{a/4(r^2+s^2)} \left\{\frac{(-1)^{a+1}q^{a/4(s+1)+\frac{s+1}{2}}Bp(x)}{2(1-q)} J_{r,s+1}\left(x, q^{a+2/4}y, a, A, B; q\right) + \frac{q^{a/4-\frac{as}{4}}Bp(x)}{2(1-q)} J_{r,s-1}\left(x, q^{a/4}y, a, A, B; q\right)\right\} t^r w^s$$

$$= \sum_{r,s=-\infty}^{\infty} q^{a/4(r^2+s^2)} D_{y,q} J_{r,s}(x,y,a,A,B;q) t^r w^s. \quad (3.4)$$

By equating the coefficients of $t^r w^s$ in the above equation, we get

$$D_{y,q} J_{r,s}(x,y,a,A,B;q)$$

$$= \frac{(-1)^{a+1}q^{a/4(s+1)+\frac{s+1}{2}}Bp(x)}{2(1-q)} J_{r,s+1}\left(x, q^{a+2/4}y, a, A, B; q\right) + \frac{q^{a/4(1-s)}Bp(x)}{2(1-q)} J_{r,s-1}\left(x, q^{a/4}y, a, A, B; q\right)$$

Hence the q-difference equation for $J_{r,s}(x,y,a,A,B;q)$ is given by

$$D_{y,q} J_{r,s}(x,y,a,A,B;q) = \frac{q^{a/4}Bp(x)}{2(1-q)}$$

$$\times\left\{(-1)^{a+1}q^{\frac{as}{4}+\frac{s+1}{2}} J_{r,s+1}\left(x, q^{a+2/4}y, a, A, B; q\right) + q^{-\frac{as}{4}} J_{r,s-1}\left(x, q^{a/4}y, a, A, B; q\right)\right\} . \quad (3.5)$$

4. The recurrence relations of the matrix function $J_{r,s}(x,y,a,A,B;q)$

The following q-recurrence relations of the function $J_{r,s}(x,y,a,A,B;q)$ holds true:

Theorem 4.1.

$$J_{r,s}(x,y,a,A,B;q) = \frac{2[\sqrt{2A}]^{-1}}{x}(1-q^{r+1})\left(q^{a/4}\right)^{r+1} J_{r+1,s}\left(q^{-a/4}x, y, a, A, B; q\right)$$

$$+(-1)^{(a+1)} q^{\frac{a+2}{2}(r+1)} J_{r+2,s}(x,y,a,A,B;q)$$

Proof. From (2.2), we know that

$$J_{r,s}(x,y,a,A,B;q)=\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_r(q;q)_s}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r)+n(n+s)]}(q^{r+m+1};q)}{(q^{1+r};q)_m(q^{r+m+1};q)(q;q)_m(q^{1+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

By using relations (1.9) and (1.11), we get

$$J_{r,s}(x,y,a,A,B;q)=\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_r(q;q)_s}\sum_{m,n=0}^{\infty}(-1)^{(m+n)(a+1)}$$

$$\times\frac{q^{a/2[m(m+r)+n(n+s)]}(1-q^{r+1}+q^{r+1}-q^{r+m+1})}{(q^{1+r};q)_{m+1}(q;q)_m(q^{1+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

Using (1.11), we get

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[\sqrt{2A}]^{-1}}{x}(1-q^{r+1})\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r+1}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_r(q;q)_s}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r+1)+n(n+s)]-\frac{a}{2}m}}{(q^{1+r};q)_1(q^{2+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

$$+q^{r+1}\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r+2}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_r(q;q)_s}$$

$$\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r)+n(n+s)]}(1-q^m)}{(q^{1+r};q)_1(q^{2+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n}\left[\left(\frac{x\sqrt{2A}}{2}\right)^{2}\right]^{m-1}\left(\frac{Byp(x)}{2}\right)^{2n}$$

Using, $(q;q)_m=(1-q^m)(q;q)_{m-1}$ and replace m by $r+1$ in r.h.s. in a second term, we obtain

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[\sqrt{2A}]^{-1}}{x}(1-q^{r+1})\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r+1}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_{r+1}(q;q)_s}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r+1)+n(n+s)]}}{(q^{2+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n}\left(\frac{q^{-a/4}x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

$$+q^{\frac{a+2}{2}(r+1)}\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r+2}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_{r+2}(q;q)_s}$$

$$\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n+1)(a+1)}q^{a/2[m(m+r+2)+n(n+s)]}}{(q^{3+r};q)_m(q;q)_m(q^{1+s};q)_n(q;q)_n}\left[\left(\frac{x\sqrt{2A}}{2}\right)^{2}\right]^{m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[\sqrt{2A}]^{-1}}{x}(1-q^{r+1})\left(q^{a/4}\right)^{r+1}J_{r+1,s}(q^{-a/4}x,y,a,A,B;q)$$

$$+(-1)^{(a+1)}q^{\frac{a+2}{2}(r+1)}J_{r+2,s}(x,y,a,A,B;q)$$

Which the required relation.

Similarly, if we write $(1-q^m+q^m-q^{r+m+1})$ instead of $(1-q^{r+1}+q^{r+1}-q^{r+m+1})$, we prove the following lemma.

corollary 4.1.

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[\sqrt{2A}]^{-1}}{x}(1-q^{r+1})\left(q^{\frac{a-2}{2}}\right)^{r+1}J_{r+1,s}\left(q^{\frac{2-a}{4}}x,y,a,A,B;q\right)$$

$$+(-1)^{(a+1)}q^{\frac{a(r+1)}{2}}J_{r+2,s}(x,y,a,A,B;q)$$

Theorem 4.2.

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[B]^{-1}}{yp(x)}(1-q^{s+1})\left(q^{a/4}\right)^{s+1}J_{r,s+1}(x,q^{-a/4}y,a,A,B;q)$$

$$+(-1)^{(a+1)}q^{\frac{a+2}{2}(s+1)}J_{r,s+2}(x,y,a,A,B;q)\ .$$

Proof. From (2.1), we know that

$$J_{r,s}(x,y,a,A,B;q)=\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s}}{(q;q)_r(q;q)_s}\sum_{m,n=0}^{\infty}(-1)^{(m+n)(a+1)}$$

$$\times\frac{q^{a/2[m(m+r)+n(n+s)]}(1-q^{s+1}+q^{s+1}-q^{s+n+1})}{(q^{1+r};q)_m(q;q)_m(q^{1+s};q)_{n+1}(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

Using (1.11), we get

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[B]^{-1}}{yp(x)}(1-q^{s+1})\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s+1}}{(q;q)_r(q;q)_s}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r)+n(n+s+1)]-\frac{a}{2}n}}{(q^{1+r};q)_m(q;q)_m(q^{1+s};q)_1(q^{2+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

$$+q^{s+1}\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s+2}}{(q;q)_r(q;q)_s}$$

$$\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r)+n(n+s)]}(1-q^n)}{(q^{1+r};q)_m(q;q)_m(q^{1+s};q)_1(q^{2+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\left(\frac{Byp(x)}{2}\right)^{2}\right)^{n-1}$$

Using, $(q;q)_n=(1-q^n)(q;q)_{n-1}$ and replace n by $n+1$ in r.h.s. in a second term, we obtain

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[B]^{-1}}{yp(x)}(1-q^{s+1})\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s+1}}{(q;q)_r(q;q)_{s+1}}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n)(a+1)}q^{a/2[m(m+r)+n(n+s+1)]}}{(q^{1+r};q)_m(q;q)_m(q^{2+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{q^{-a/4}Byp(x)}{2}\right)^{2n}$$

$$+q^{\frac{a+2}{2}(s+1)}\frac{\left(\frac{x\sqrt{2A}}{2}\right)^{r}\left(\frac{Byp(x)}{2}\right)^{s+2}}{(q;q)_r(q;q)_{s+2}}$$

$$\sum_{m,n=0}^{\infty}\frac{(-1)^{(m+n+1)(a+1)}q^{a/2[m(m+r)+n(n+s+2)]}}{(q^{1+r};q)_m(q;q)_m(q^{3+s};q)_n(q;q)_n}\left(\frac{x\sqrt{2A}}{2}\right)^{2m}\left(\frac{Byp(x)}{2}\right)^{2n}$$

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[B]^{-1}}{yp(x)}(1-q^{s+1})\left(q^{a/4}\right)^{s+1}J_{r,s+1}(x,q^{-a/4}y,a,A,B;q)$$

$$+(-1)^{(a+1)}q^{\frac{a+2}{2}(s+1)}J_{r,s+2}(x,y,a,A,B;q)$$

Which the required relation.

Similarly, if we write $(1-q^n+q^n-q^{s+n+1})$ instead of $(1-q^{s+1}+q^{s+1}-q^{s+n+1})$, we prove the following lemma.

corollary 4.2.

$$J_{r,s}(x,y,a,A,B;q)=\frac{2[B]^{-1}}{yp(x)}(1-q^{s+1})\left(q^{\frac{a-2}{2}}\right)^{s+1}J_{r,s+1}\left(x,q^{\frac{2-a}{4}}y,a,A,B;q\right)$$

$$+(-1)^{(a+1)}q^{\frac{a}{2}(s+1)}J_{r,s+2}(x,y,a,A,B;q).$$

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مقالة بحثية

دالة مصفوفة بسل الأساسية المعمّمة ذات متغيرين

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المُلخّص

دالة بسل تعتبر من أفضل الدوال الخاصة المعروفة، ضمن الرياضيات البحتة والتطبيقية. في هذه الورقة قدمنا دالة مصفوفة بسل (Bessel) الأساسية المعمّمة ذات متغيرين ثم دراستها واشتقاق خصائصها الأساسية مثل التمثيلات فوق الهندسية الأساسية لدالة المولدة والمعادلة التفاضلية الأساسية وكذلك العلاقات التكرارية.

الكلمات الرئيسية: دالة مصفوفة بسل الأساسية المعمّمة ذات متغيرين، الدالة المولدة، المعادلة التفاضلية الأساسية، العلاقات التكرارية.

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