Electronic Journal of University of Aden for Basic and Applied Sciences
EJUA-BA Vol. 1 No. 4 (2020)
https://doi.org/10.47372/ejua-ba.2020.4.62
ISSN: 2708-0684

RESEARCH ARTICLE

ON α-MINIMAL AND α-MAXIMAL SETS IN TOPOLOGICAL SPACES

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Received: 05 November 2020 / Accepted: 24 December 2020 / Published online: 30 December 2020

Abstract

In this paper, we introduce and study new types of sets called α-minimal open and α-maximal open sets and study some of their fundamental properties, furthermore, relation shapes between there concepts and some known concepts are studied. Finally, some counter examples are proved.

Keywords: α-open, Maximal open, Minimal open, Maximal closed and minimal closed sets.

1. Introduction

The concepts of minimal open sets with its applications were introduced in (2001) by Nakaoka & Oda [1]. A maximal open set in topological spaces was introduced and their properties were studied in(2003) by Nakaoka & Oda [2]. More precisely with the characterization of minimal open sets it is proved that any subset of a minimal open set is pre-open. By the dual concepts of minimal open sets and maximal open sets in (2006), Nakaoka & Oda [3] introduced the concepts of minimal closed sets and maximal closed sets and obtained the results by dualizing the known results regarding minimal open sets and maximal open sets.

Several authors have used these new notions in many directions [4, 5]. For instance, Benchalli [6, 7] introduced the new concepts of generalized minimal closed sets.
The concepts of maximal and minimal mean open sets are introduced and considered by Bagchi and Mukharjee [8]. Moreover, maximal and minimal Θ-open sets and their properties are considered by M. Caldas, Jafari and Moshokoa [9]. The concept of minimal ɤ-open sets are introduced and considered by Hussain and Ahmad [10].

2. Preliminaries

Definition 2.1[1]. A proper nonempty open set U of X is said to be a minimal open set if any open set which contained in U is Ø or U.

Definition 2.2[2]. A proper nonempty open set U of X is said to be a maximal open set if any open set which contains U is X or U.

Definition 2.3[3]. A proper nonempty closed subset F of X is said to be a minimal closed set if any closed set which contained in F is Ø or F.

Definition 2.4[3]. A proper nonempty closed subset F of X is said to be a maximal closed set if any closed set which contains F is X or F.

Lemma 2.5[6]. If $Y \subseteq X$ is any subspace of a topological space (X, τ) and U is any minimal open set in X then $Y \cap U$ is a minimal open set in Y.

Definition 2.6[11]. A subset A of a space X is said to be α-open set if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The complement of an α-open set is said to be α-closed. As in the usual sense, the intersection of all α-closed sets of X containing A is called the α-closure of A. also the union of all α-open sets of X contained in A is called the α-interior of A.

3. α-minimal open and α-maximal closed sets

In this section, we study and investigate classes of sets called α-minimal open set and α-maximal closed set and study some of their fundamental properties

Definition 3.1. A set A in a topological space X is said to be α-minimal open set if there exists a minimal open set M Such that $M \subset A \subset \mathrm{Int}(\mathrm{Cl}(M))$.

The family of all α-minimal open sets in a topological space X is denoted by $\alpha\, M_i O(X)$.

Remark 3.2. Note that every Minimal open set is α-minimal open set by Definition 3.1 put $A = M$. But not the converse which is shown by the following example.

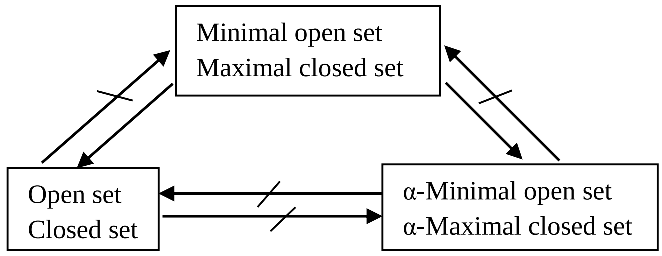
Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ be topological spaces. In τ closed sets are: X, $\{b, c\}$, $\{c\}$, $\emptyset$, Minimal open set is: $\{a\}$. α-Minimal open sets: $\{a\}$, $\{a, b\}$, $\{a, c\}$.
$\{a, c\} \in \alpha\, M_iO(X)$ but it $\notin M_iO(X)$, then $\alpha\, M_iO(X) \not\subset M_iO(X)$, also $\{a, c\}$ is not open set .
In σ closed sets are: X, $\{b, c\}$, $\{a, c\}$, $\{c\}$, $\emptyset$, Minimal open set are $\{a\}$, $\{b\}$. α-Minimal open sets are $\{a\}$,$\{b\}$. $\{a, b\}$ is open set but $\notin M_iO(X)$, also $\{a, b\} \notin \alpha\, M_iO(X)$.

Definition 3.4. A subset N of a topological space X is said to be α-maximal closed set if $X - N$ is α-minimal open set. The family of all α-maximal closed sets in a topological space X is denoted by $\alpha M_aC(X)$.

Remark 3.5. Every maximal closed set is α-maximal closed set by Remark 3.2 and Definition 3.4. But not converse which is shown by the following example.

Example 3.6. From Example 3.3 in τ closed sets are: $\{X, \emptyset, \{b, c\}, \{c\}\}$.
Maximal Closed set is: $\{b, c\}$. α-maximal closed sets:$\{\{b\}, \{c\}, \{b, c\}\}$.
$\{b\} \in \alpha M_aC(X)$ but it $\notin M_aC(X)$, then $\alpha M_aC(X) \not\subset M_aC(X)$, also$\{b\}$ is not closed set.
In σ closed sets are: X, $\{b, c\}$, $\{a, c\}$, $\{c\}$, $\emptyset$, Maximal closed set are $\{b, c\}$, $\{a, c\}$.
α-Maximal closed sets:$\{b, c\}$, $\{a, c\}$.
$\{c\}$ is closed set but $\notin M_aC(X)$, also $\{c\} \notin \alpha M_aC(X)$.

The above results are given in below implication diagram.



Theorem 3.7. Every α-minimal open set in a topological space (X, τ) is α-open set.

Proof. Let A be any α-minimal open set in a topological space (X, τ). By the Definition 3.1 $U \subset A \subset Int(Cl(U))$, And U is a minimal open set in X. But every minimal open set is an open set in X. Therefore
$U \subset A \subset Int(Cl(U)) = Int(Cl(Int U))$.
Hence A is an α-open set.

Remark 3.8. The converse of the Theorem above need not be true.

Example 3.9. Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}, X\}$, closed sets: $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{b, c, d\}$, X. Minimal open sets: $\{a\}$,$\{c, d\}$, α-open sets: $\emptyset$, $\{a\}$, $\{c, d\}$, $\{b, c, d\}$, $\{a, c, d\}$, X, α-minimal open sets : $\{a\}$, $\{c, d\}$, $\{b, c, d\}$.

Theorem 3.10. Let X be a topological space .Y be subspace of X and M be a subset of Y .If M is α-minimal open in X then M is α-minimal open in Y.

Proof. Suppose M is α-minimal open in X. By definition 3.1 there exists a minimal open set N in X such that $N \subset M \subset Int(Cl(N))$. Now $N \subset Y$. Hence $Y \cap N = N$. Since N is minimal open in X. $Y \cap N = N$ is minimal open in Y by (lemma 2.5) Now we have $N \subset M \subset Int(Cl(N))$. Therefore $Y \cap N \subset Y \cap M \subset Y \cap Int(Cl(N))$, which implies $N \subset M \subset IntCl(N)$. Thus there exists a minimal open set N in Y Such that $N \subset M \subset IntCl(N)$. Therefore by definition 3.1 it follows that M is α-minimal open in Y.

Theorem 3.11. A subset W of a topological space X is α-maximal closed iff there exists a maximal closed set N in X such that $Cl(Int\,(N) \subset W \subset N$.

Proof. Suppose W is an α-maximal closed in X then by definition 3.4 then $X - W$ is α-minimal open in X. Therefore by Definition 3.1 there exists a minimal open set M such that $M \subset X - W \subset IntCl(M)$ which implies **that** $X - IntCl(M) \subset X - [X - W] \subset X - M$ which implies $X - IntCl(M) \subset W \subset X - M$. But it is know that $X - Int\big(Cl(M)\big) = Cl(X - Cl\, M) = Cl(Int(\, X - M))$ take $X - M = N$ so, that N is a maximal closed set such that $Cl(Int(N)) \subset W \subset N$.

Conversely, suppose that there exist a maximal closed set N in X such that $Cl(Int\,(N) \subset W \subset N$. Therefore it follows that $X - N \subset [X - W] \subset X - Cl(Int\,(N)$. But it is know that $X - Cl\big(Int\,(N)\big) = Int\big(X - Int(N)\big) = Int(Cl(X - N))$. Therefore there exists a minimal open set $X - N$ such that $X - N \subset X - W \subset IntCl(X - N)$. Thus by Definition 3.1 it follows that $X - W$ is α-minimal open in X. Hence by Definition 3.4 it follows that W is α-maximal closed set.

Theorem 3.12. Every α-maximal closed set in a topological space (X, τ) is an α-closed.

Proof. By Definition 3.4 and Theorem 3.11.

4. α-maximal open and α-minimal closed sets

In this section, we study and investigate classes of open sets called α-maximal open set and α-minimal closed set and study some of their fundamental properties**.**

Definition 4.1. A set A in a topological space X is said to be α-maximal open set if there exists a maximal open set M Such that $M \subset A \subset Int(Cl(M))$. The family of all α-maximal open sets in a topological space X is denoted by $\alpha\, M_aO(X)$.

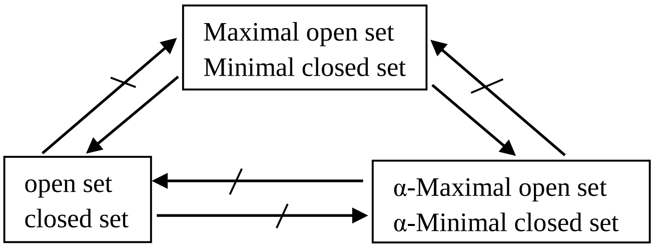
Remark 4.2. Every Maximal open set is α-maximal open set by Definition 4.1. put $A = M$. But not the converse which is shown by the following example.

Example 4.3. Let X = {a, b, c, d, e}, τ ={X, Ø, {a}, {b}, {a, b}, {a, b, c}} be a topology on X. closed sets are: {X, Ø, {b, c, d, e}, {a, c, d, e},{c, d, e},{d, e}}
Maximal open set is{a, b, c}.α-maximal open sets {a, b, c}, {a, b, c, d},{a, b, c, e}.
{a, b, c, e} $\in \alpha M_a O(X)$, but it is not maximal open set, then $\alpha M_a O(X) \not\subset M_a O(X)$.
Also {a, b, c, e}is not open set . {a, b} is open set but is not maximal open or is not α-maximal open set.

Definition 4.4. A subset N of a topological space X is said to be α-minimal closed set if X−N is α-maximal open set. The family of all α-minimal closed sets in a topological space X is denoted by $\alpha M_i C(X)$.

Remark 4.5. Every minimal closed set is α-minimal closed set by Remark 4.2 and Definition 4.4. But not converse which is shown by the following example.

Example 4.6. Let X = {a, b, c, d, e}, τ = {X, Ø, {a}, {b}, {a, b}, {a, b, c}} be a topology on X. Closed sets are X, Ø, {b, c, d, e}, {a, c,d, e}, {c, d, e}, {d, e}
Minimal Closed set: {d, e}. α-minimal closed sets: {e}, {d}, {d, e}. Since {e} is α-minimal closed set but it is not minimal closed, then $\alpha M_i C(X) \not\subset M_i C(X)$, and {e} is not closed set. Also{c, d, e} is closed but it is not minimal closed or α-minimal closed set.
The above results are given in below implication diagram.



Theorem 4.7. Every α-maximal open set in a topological space (X, τ) is an α-open set.

Proof. Let A be any α-maximal open set in a topological space (X, τ). By the Definition 4.1 $U \subset A \subset \mathrm{Int}(\mathrm{Cl}(U))$, where U is a maximal open set in X. But every maximal open set is an open set in X. Therefore $U \subset A \subset \mathrm{Int}(\mathrm{Cl}(U))$. and U is an open in X. Hence A is an α-open set.

Remark 4.8. The converse of the Theorem above need not be true.

Example 4.9. By Example 4.3 α-open sets are :
X,Ø,{a},{b},{a ,b},{a, b, c},{a, b, c ,d} ,{a ,b ,c, e} .

Theorem 4.10. Let X be a topological space .Y be subspace of X and M be a subset of Y. If M is α-maximal open in X then M is α-maximal open in Y.

Proof. Suppose M is α-maximal open in X. By Definition 4.1 there exists a maximal open set N in X such that N⊂M⊂Int(Cl(N)). Now N⊂Y. Hence Y∩ N=N. Since N is maximal open in X. Y∩ N=N is maximal open in Y. Now we have $N \subset M \subset \mathrm{Int}(\mathrm{Cl}(N))$. Therefore $Y \cap N \subset Y \cap M \subset Y \cap \mathrm{Int}(\mathrm{Cl}(N))$, which implies $N \subset M \subset \mathrm{IntCl}(N)$. Thus there exists a maximal open set N in Y Such that $N \subset M \subset \mathrm{IntCl}(N)$. Therefore by Definition 4.1 it follows that M is α-maximal open in Y.

Theorem 4.11. A subset W of a topological space X is α-minimal closed iff there exists a minimal closed set N in X such that $\mathrm{Cl}(\mathrm{Int}\ (N)) \subset W \subset N$.

Proof. Suppose W is an α-minimal closed in X then by Definition 4.4 $X - W$ is α-maximal open in X. Therefore by Definition 4.1 there exists a maximal open set M such that $M \subset X - W \subset \mathrm{IntCl}(M)$ which implies **that** $X - \mathrm{IntCl}(M) \subset X - [X - W] \subset X - M$ which implies $X - \mathrm{IntCl}(M) \subset W \subset X - M$. But it is know that: $X - \mathrm{Int}(\mathrm{Cl}(M)) = \mathrm{Cl}(X - \mathrm{Cl}\ M) = \mathrm{Cl}(\mathrm{Int}(X - M))$ take $X - M = N$ so, that N is a minimal closed set such that $\mathrm{Cl}(\mathrm{Int}(N)) \subset W \subset N$.

Conversely, suppose that there exist a minimal closed set N in X such that $\mathrm{Cl}(\mathrm{Int}\ (N) \subset W \subset N$. Therefore it follows that $X - N \subset [X - W] \subset X - \mathrm{Cl}(\mathrm{Int}\ (N)$. But it is know that $X - \mathrm{Cl}(\mathrm{Int}\ (N)) = \mathrm{Int}(X - \mathrm{Int}(N)) = \mathrm{Int}(\mathrm{Cl}(X - N))$. Therefore there exists a maximal open set $X - N$ such that $X - N \subset X - W \subset \mathrm{IntCl}(X - N)$. Thus by Definition 4.1. It follows that $X - W$ is α-maximal open in X. Hence by Definition 4.4. it follows that W is α-minimal closed set.

Theorem 4.12. Every α-minimal closed set in a topological space (X, τ) is an α- closed.

Proof. By Definition 4.4 and Theorem 4.11.

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مقالة بحثية

حول مجموعات الالفا الصغرى والعظمى في الفراغات الطوبولوجية

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استلم في: 05 نوفمبر 2020/ قبل في: 24 ديسمبر 2020 / نشر في: 30 ديسمبر 2020

المُلخّص

ناقش هذا البحث انواع جديدة من المجموعات. وكل المفاهيم تتم دارستها بالاعتماد على مفهوم المجموعات من النمط الفا، قمنا بتعريف نمطين جديدين من المجموعات الالفا المفتوحة هما α-Minimal و α-Maximal، وتمت دراسة الخواص الاساسية للمفاهيم المطروحة في هذا البحث. اضافة الى ذلك تمت دراسة أشكال العلاقة بين المفاهيم وبعض المفاهيم المعروفة. أخيرًا تم إثبات بعض الأمثلة المعاكسة.

الكلمات الرئيسية: المجموعات الفا المفتوحة، الصغرى المفتوحة، العظمى المفتوحة، الصغرى و العظمى المغلقة.

How to cite this article:

K. M. A. AL Hamadi and E. Q. A. Mohammed, “On A-Minimal and A-Maximal Sets in Topological Spaces”, *Electron. J. Univ. Aden Basic Appl. Sci.*, vol. 1, no. 4, pp. 247-250, Dec. 2020. DOI: 10.47372/ejua-ba.2020.4.62