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RESEARCH ARTICLE

CERTAIN EXTENSION OF THE HURWITZ-LERCH ZETA FUNCTION AND ITS PROPERTIES

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Abstract

The main object of this paper is to introduce a new extension of Hurwitz-Lerch Zeta function by using the extended Beta function given in [1]. Some recurrence relations, generating functions and integral representations are derived for that new extension.

Keywords: Extended Beta function, extended Hurwitz-Lerch Zeta function, recurrence relation, generating functions, integral representation.

1. Introduction

Very recently, Al-Gonah and Mohammed **[1]**, introduced the extended Beta function and studied some properties of that function. The extended Beta function is defined by **[1,p.259(2.2)]**:

$$B_p^{(\rho,\sigma,\tau)}(x,y)$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} E_{\rho,\sigma}^{\tau}\left(\frac{-p}{t(1-t)}\right) dt, \qquad (1.1)$$

$$(Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0, Re(x) > 0, Re(y) > 0),$$

where $E_{\rho,\sigma}^{\tau}(z)$ denotes the generalized Mittag-Leffler function defined by **[12,p.7(1.3)]** see also **[13]**:

$$E_{\rho,\sigma}^{\tau}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\rho n+\sigma)} \frac{z^n}{n!}, \qquad (1.2)$$

$$(z,\rho,\sigma,\tau \in \mathbb{C}; Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0).$$

It is clear that

$$\Gamma(\sigma) E_{1,\sigma}^{\tau}(z) = {}_1F_1(\tau;\sigma;z), \qquad (1.3a)$$

$$E_{1,1}^{1}(z) = e^z. \qquad (1.3b)$$

Note that, by using relations (1.3a) and (1.3b), we get the following special cases:

$$B_p^{(1,\sigma,\tau)}(x,y) = \frac{1}{\Gamma(\sigma)} B_p^{(\tau,\sigma)}(x,y), \qquad (1.4a)$$

$$B_p^{(1,1,1)}(x,y) = B\ (x,y;p), \qquad (1.4b)$$

Where $B_p^{(\tau,\sigma)}(x,y)$ and $B\ (x,y;p)$ denote the extended Beta functions defined **[9,p.4602(2)]** and **[4,p.20(1.7)]** by

$$B_p^{(\rho,\sigma)}(x,y)$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\rho;\sigma;\frac{-p}{t(1-t)}\right) dt, \qquad (1.5)$$

$$(Re(p) \geq 0; Re(\rho) > 0, Re(\sigma) > 0, Re(x) > 0, Re(y) > 0),$$

$$B\ (x,y;p)$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \qquad (1.6)$$

$$(Re(p) \geq 0, Re(x) > 0, Re(y) > 0),$$

respectively.

The extended Beta function given in equation (1.1) is used in **[2]** to define a new form of the extended Gauss hypergeometric function as:

$$F_p^{(\rho,\sigma,\tau)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\rho,\sigma,\tau)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (1.7)$$

$(Re(p) \geq 0, |z| < 1; Re(c) > Re(b) > 0, Re(\rho) > 0\ Re(\sigma) > 0, Re(\tau) > 0).$

Note that

$$F_p^{(1,\sigma,\tau)}(a,b;c;z) = \frac{1}{\Gamma(\sigma)} F_p^{(\tau,\sigma)}(a,b;c;z), \quad (1.8a)$$

$$F_p^{(1,1,1)}(a,b;c;z) = F_p\ (a,b;c;z), \quad (1.8b)$$

$$F_0^{(\rho,1,\tau)}(a,b;c;z) = {}_2F_1(a,b;c;z), \quad (1.8c)$$

Where $F_p^{(\tau,\sigma)}(a,b;c;z)$ and $F_p\ (a,b;c;z)$ denoted the extended forms of hypergeometric functions given in **[9,p.4606]** and **[5,p.591(2.2)]** respectively.

Various forms of generalizations of the Hurwitz-Lerch Zeta function$\Phi\ (z,s,a)$ given in **[14, p.121]**) have been considered by many authors, see **[3,6-8,10,11]**. Some interesting forms of generalized Hurwitz-Lerch Zeta functions are defined in **[8,p.100]** and **[7, p. 313]** as:

$$\Phi_\mu^*\ (z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s}, \quad (1.9)$$

$(\mu \in \mathbb{C}; a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}$ when $|z| < 1, Re(s-\mu) > 1$ when $|z| = 1),$

and

$$\Phi_{\lambda,\mu;\nu}\ (z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n n!}\frac{z^n}{(n+a)^s}, \quad (1.10)$$

$(\lambda,\mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}$ when $|z| < 1,$ $Re(s+\nu-\lambda-\mu) > 1$ when $|z| = 1),$

respectively.

Recently, several forms of extended Beta functions are used to investigate and introduce new forms of extended Zeta function given in equation (1.10). Parmar and Raina **[11]** used the extended Beta function $B\ (x,y;p)$ to introduce the following extension of generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\nu}\ (z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B(\mu+n,\nu-\mu;p)}{B(\mu,\nu-\mu)}\frac{z^n}{(n+a)^s}, \quad (1.11)$$

$(\ p \geq 0; \lambda,\mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}$ when $|z| < 1; Re(s+\nu-\lambda-\mu) > 1$ when $|z| = 1),$

with the integral representation **[11,p.120(3.1)]**:

$$\Phi_{\lambda,\mu;\nu}\ (z,s,a;p) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-at} F_p(\lambda,\mu;\nu;ze^{-t})\, dt, \quad (1.12)$$

$(Re(p) \geq 0; p = 0, Re(a) > 0; Re(s) > 0$ when $|z| \leq 1\ (z \neq 1); Re(s) > 1$ when $z = 1).$

Very recently, Parmar *et al*. **[10]** used the extended Beta function$B_p^{(\rho,\sigma)}(x,y)$ to introduce the following extension of generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}\ (z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{(n+a)^s}, \quad (1.13)$$

$(\ p \geq 0, Re(\rho) > 0, Re(\sigma) > 0; \lambda,\mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-;$ $s \in \mathbb{C}$ when $|z| < 1; Re(s+\nu-\lambda-\mu) > 1$ when $|z| = 1),$

with the integral representation **[10,p.180(3.1)]**:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma)}\ (z,s,a;p) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-at} F_p^{(\rho,\sigma)}(\lambda,\mu;\nu;ze^{-t})\, dt, \quad (1.14)$$

$(\ Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0; p = 0, Re(a) > 0$ $; Re(s) > 0$ when $|z| \leq 1\ (z \neq 1); Re(s) > 1$ when $z = 1).$

Motivated by various recent interesting extensions of the Hurwitz-Lerch Zeta function, we introduce a new form of extended Hurwitz-Lerch Zeta function and investigate its properties.

2. A new extended Hurwitz-Lerch Zeta function

In this section, we use the extended Beta function$B_p^{(\rho,\sigma,\tau)}(x,y)$ defined by equation (1.1) to propose a new extended Hurwitz-Lerch Zeta function as follows:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}\ (z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{(n+a)^s}, \quad (2.1)$$

$(Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0; \lambda,\mu \in \mathbb{C}$ $; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}$ when $|z| < 1; Re(s+\nu-\lambda-\mu)$ > 1 when $|z| = 1).$

We observe that

$$\Phi_{\lambda,\mu;\nu}^{(1,1,1)}\ (z,s,a;0) = \Phi_{\lambda,\mu;\nu}\ (z,s,a), \quad (2.2a)$$

$$\Phi_{\lambda,\mu;\nu}^{(1,\sigma,\tau)}\ (z,s,a;p) = \frac{1}{\Gamma(\sigma)}\Phi_{\lambda,\mu;\nu}^{(\tau,\sigma)}\ (z,s,a;p), \quad (2.2b)$$

$$\Phi_{\lambda,\mu;\nu}^{(1,1,1)}(z,s,a;p) = \Phi_{\lambda,\mu;\nu}(z,s,a;p), \qquad (2.2c)$$

$$\Phi_{1,\mu;1}^{(1,1,1)}(z,s,a;p) = \Phi_{\mu}^{*}(z,s,a;p), \qquad (2.2d)$$

$$\Phi_{1,\mu;1}^{(1,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\sigma)}\Phi_{\mu}^{*(\tau,\sigma)}(z,s,a;p), \qquad (2.2e)$$

where $\Phi_{\lambda,\mu;\nu}(z,s,a)$, $\Phi_{\lambda,\mu;\nu}^{(\tau,\sigma)}(z,s,a;p)$ and $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ are defined by equations (1.10), (1.12) and (1.11) respectively, $\Phi_{\mu}^{*}(z,s,a;p)$ and $\Phi_{\mu}^{*(\tau,\sigma)}(z,s,a;p)$ are defined in [11] and [10] as:

$$\Phi_{\mu}^{*}(z,s,a;p) = \sum_{n=0}^{\infty}\frac{B(\mu+n,1-\mu;p)}{B(\mu,1-\mu)}\frac{z^n}{(n+a)^s}, \qquad (2.3)$$

$$(p \geq 0; \mu \in \mathbb{C}; a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1, Re(s-\mu) > 1 \text{ when} |z| = 1),$$

$$\Phi_{\mu}^{*(\tau,\sigma)}(z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\tau,\sigma)}(\mu+n,1-\mu)}{B(\mu,1-\mu)}\frac{z^n}{(n+a)^s}, \qquad (2.4)$$

$$(p \geq 0, Re(\tau) > 0, Re(\sigma) > 0; \mu \in \mathbb{C}; a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1, Re(s-\mu) > 1 \text{ when} |z| = 1),$$

respectively.

Remark2.1. The extended Hurwitz-Lerch function$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$ defined by equation (2.1) is seen to satisfy the following limit case:

$$\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(z,s,a;p) = \lim_{|\lambda|\to\infty}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}\left(\frac{z}{\lambda},s,a;p\right)\right\},$$

$$= \sum_{n=0}^{\infty}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{n!\,(n+a)^s}, \qquad (2.5)$$

$$(Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0; \mu \in \mathbb{C}; \nu, a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; Re(s+\nu-\mu) > 1 \text{ when} |z| = 1).$$

Note that

$$\Phi_{\mu;1}^{*(1,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\sigma)}\Phi_{\mu}^{*(\tau,\sigma)}(z,s,a;p), \quad (2.6)$$

Where $\Phi_{\mu}^{*(\tau,\sigma)}(z,s,a;p)$ is defined by equation (2.4).

Some properties of the extended Hurwitz-Lerch Zeta function$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$are established in the form of the following theorems:

Theorem 2.1.*For the extended Hurwitz-Lerch Zeta function$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, we have the following recurrence relation:*

$$\nu\,\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)} = \mu\,\Phi_{\lambda,\mu+1;\nu+1}^{(\rho,\sigma,\tau)}(z,s,a;p) + (\nu-\mu)\,\Phi_{\lambda,\mu;\nu+1}^{(\rho,\sigma,\tau)}(z,s,a;p), \qquad (2.7)$$

$$(Re(p) \geq 0, Re(\nu) > Re(\mu) > 0).$$

Proof. Using the following known relation **[1,p.259(2.5)]**:

$$B_p^{(\rho,\sigma,\tau)}(x,y) = B_p^{(\rho,\sigma,\tau)}(x+1,y) + B_p^{(\rho,\sigma,\tau)}(x,y+1), \qquad (2.8)$$

in definition (2.1), we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s} \times \left[\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n+1,\nu-\mu) + B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu+1)}{B(\mu,\nu-\mu)}\right]$$

$$= \frac{B(\mu+1,\nu-\mu)}{B(\mu,\nu-\mu)} \times \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n+1,\nu-\mu)}{B(\mu+1,\nu-\mu)}\frac{z^n}{(n+a)^s}$$

$$+ \frac{B(\mu,\nu-\mu+1)}{B(\mu,\nu-\mu)} \times \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu+1)}{B(\mu,\nu-\mu+1)}\frac{z^n}{(n+a)^s}, \quad (2.9)$$

which on using the relation [15]:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad (x,y \in \mathbb{C}\backslash\mathbb{Z}_0^-), \qquad (2.10)$$

and then using definition (2.1) gives the desired result.

Remark 2.2. Using the following relations [1]:

$$(1+\rho\tau-\sigma)B_p^{(\rho,\sigma,\tau)}(x,y) = \rho\,\tau\,B_p^{(\rho,\sigma,\tau+1)}(x,y) - B_p^{(\rho,\sigma-1,\tau)}(x,y), \qquad (2.11)$$

$$p\, B_p^{(\rho,\sigma,\tau)}(x-1,y-1)$$

$$= B_p^{(\rho,\sigma-\rho,\tau-1)}(x,y) - B_p^{(\rho,\sigma-\rho,\tau)}(x,y), \qquad (2.12)$$

and proceeding on the same lines of proof of Theorem 2.1, we get some recurrence relations in the form of the following theorem:

Theorem 2.2. *For the extended Hurwitz-LerchZetafunction* $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, *we have the following recurrence relations:*

$$(1+\rho\tau-\sigma)\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$= \rho\tau\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau+1)}(z,s,a;p) - \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma-1,\tau)}(z,s,a;p), \qquad (2.13)$$

$$(Re(p)\geq 0, Re(\nu) > Re(\mu) > 0, Re(\rho) > 0, Re(\tau) > 0, Re(\sigma) > 1),$$

$$p\nu\,(\nu+1)\,\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$= \mu(\nu-\mu)\left\{\Phi_{\lambda,\mu+1;\nu+2}^{(\rho,\sigma-\rho,\tau-1)}(z,s,a;p) - \Phi_{\lambda,\mu+1;\nu+2}^{(\rho,\sigma-\rho,\tau)}(z,s,a;p)\right\}, \qquad (2.14)$$

$$(Re(p)\geq 0, Re(\nu) > Re(\mu) > 0, Re(\sigma) > Re(\rho) > 0, Re(\tau) > 1).$$

Remark 2.3. Using the following relations **[1,p.260]**:

$$B_p^{(\rho,\sigma,\tau)}(x,y) = \sum_{n=0}^{\infty} B_p^{(\rho,\sigma,\tau)}(x+n,y+1), \qquad (2.15)$$

$$B_p^{(\rho,\sigma,\tau)}(x,1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\rho,\sigma,\tau)}(x+n,1), \qquad (2.16)$$

$$B_p^{(\rho,\sigma,\tau)}(x,y) = \textstyle\sum_{n=0}^{k}\binom{k}{n} B_p^{(\rho,\sigma,\tau)}(x+n,y+k-n), \qquad (2.17)$$

and proceeding on the same lines of proof of Theorem 2.1, we get some summation relations in the form of the following theorem:

Theorem 2.3. *For the extended Hurwitz-Lerch Zeta function* $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, *we have the following summation relations:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$= (\nu-\mu)\sum_{k=0}^{\infty}\frac{(\mu)_k}{(\nu)_{k+1}}\Phi_{\lambda,\mu+k;\nu+k+1}^{(\rho,\sigma,\tau)}(z,s,a;p), \qquad (2.18)$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$= \sum_{k=0}^{\infty}\frac{(\mu-\nu+1)_k B(\mu+k,1)}{k!\,B(\mu,\nu-\mu)}\Phi_{\lambda,\mu+k;\mu+k+1}^{(\rho,\sigma,\tau)}(z,s,a;p), \qquad (2.19)$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$= \sum_{k=0}^{r}\binom{r}{k}\frac{B(\mu+k,\nu-\mu-k+r)}{B(\mu,\nu-\mu)}\Phi_{\lambda,\mu+k;\nu+r}^{(\rho,\sigma,\tau)}(z,s,a;p). \qquad (2.20)$$

Theorem 2.4. *For the extended Hurwitz-Lerch Zeta function* $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, *the following derivative formula holds true:*

$$\frac{d^k}{dz^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\right\}$$

$$= \frac{(\lambda)_k(\mu)_k}{(\nu)_k}\Phi_{\lambda+k,\mu+k;\nu+k}^{(\rho,\sigma,\tau)}(z,s,a+k;p). \qquad (2.21)$$

Proof. From definition (2.1) and using the following relation **[15]**:

$$\frac{d^k}{dz^k}z^n = \frac{n!}{(n-k)!}z^{n-k}, \qquad (k\in\mathbb{N}_0), \qquad (2.22)$$

we fined

$$\frac{d^k}{dz^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\right\}$$

$$= \sum_{n=k}^{\infty}\frac{(\lambda)_n}{(n-k)!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^{n-k}}{(n+a)^s}. \qquad (2.23)$$

Replacing n by $n+k$, we obtain

$$\frac{d^k}{dz^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(\lambda)_{n+k}}{(n)!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n+k,\nu-\mu)}{B(\mu,\nu-\mu)} \times \frac{z^n}{(n+k+a)^s}, \qquad (2.24)$$

which on using the following relations **[15]**:

$$B(b,c-b) = \frac{(c)_k}{(b)_k}B(b+k,c-b), \qquad (2.25)$$

$$(a)_{m+n} = (a)_m(a+m)_n, \qquad (2.26)$$

and then in view of definition (2.1), we get the desired result.

Theorem 2.5. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following derivative formula holds true:*

$$\frac{d^k}{dp^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\right\} = \frac{(\tau)_k(\nu-\mu)_k}{(1-\mu)_k(\nu)_{-2k}}\Phi_{\lambda,\mu-k;\nu-2k}^{(\rho,\sigma+\rho k,\tau+k)}(z,s,a;p). \quad (2.27)$$

Proof. From definition (2.1) and using relations (1.1) and (1.2), we have

$$\frac{d^k}{dp^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\right\} = \frac{1}{B(\mu,\nu-\mu)}\sum_{n=0}^{\infty}\left\{\frac{(\lambda)_n}{n!}\int_0^1 t^{\mu+n-1}(1-t)^{\nu-\mu-1}\right.$$
$$\left.\times\sum_{r=0}^{\infty}\frac{(-1)^r(\tau)_r}{\Gamma(\rho r+\sigma)(r-k)!}\frac{p^{r-k}}{t^r(1-t)^r}\frac{z^n}{(n+a)^s}\right\}dt. \quad (2.28)$$

Replacing r by $r+k$ in equation (2.28) and then using relations (1.2) and (1.1), we obtain

$$\frac{d^k}{dp^k}\left\{\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\right\}$$
$$= \frac{(-1)^k(\tau)_k B(\mu-k,\nu-\mu-k)}{B(\mu,\nu-\mu)}$$
$$\times\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_p^{(\rho,\sigma+\rho k,\tau+k)}B(\mu-k+n,\nu-\mu-k)}{B(\mu-k,\nu-\mu-k)}$$
$$\times\frac{z^n}{(n+a)^s}, \quad (2.29)$$

which on using definition (2.1) and after some simplification yields the desired result.

3. Generating relations

In this section, some generating functions for the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$ are established in the form of the following theorems:

Theorem 3.1. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following generating function holds true:*

$$\sum_{n=0}^{\infty}(\lambda)_n\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\frac{t^n}{n!}$$
$$=(1-t)^{-\lambda}\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}\left(\frac{z}{1-t},s,a;p\right), \quad (3.1)$$

$(Re(p)\geq 0, \lambda\in\mathbb{C}, |t|<1)$.

Proof. Denoting the L.H.S. of equation (3.1) by Δ then applying definition (2.1), we get

$$\Delta=\sum_{n=0}^{\infty}(\lambda)_n\left\{\sum_{k=0}^{\infty}(\lambda+n)_k\frac{B_p^{(\rho,\sigma,\tau)}B(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\right.$$
$$\left.\times\frac{z^k}{k!\,(k+a)^s}\right\}\frac{t^n}{n!}, \quad (3.2)$$

which on using relation (2.26), we obtain

$$\Delta=\sum_{k=0}^{\infty}(\lambda)_k\frac{B_p^{(\rho,\sigma,\tau)}B(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{k!\,(k+a)^s}$$
$$\times\left\{\sum_{n=0}^{\infty}\frac{(\lambda+k)_n}{n!}t^n\right\}. \quad (3.3)$$

Using the following binomial series expansion **[15]**:

$$(1-t)^{-\alpha}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}t^n, \quad (3.4)$$

for evaluating the inner sum in equation (3.3), we get the desired result.

Theorem 3.2. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following generating function holds true:*

$$\sum_{n=0}^{\infty}(s)_n\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s+n,a;p)\frac{t^n}{n!}$$
$$=\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a-t;p), \quad (3.5)$$

$(Re(p)\geq 0, \lambda\in\mathbb{C}, |t|<|a|; s\neq 1)$.

Proof. Using definition (2.1) in the R.H.S. of equation (3.5), we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a-t;p)$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a-t)^s},$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^s}\left(1-\frac{t}{(k+a)}\right)^{-s}. \quad (3.6)$$

Using relation (3.4), we obtain

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a-t;p)$$

$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^s}$$

$$\times\left\{\sum_{n=0}^{\infty}(s)_n\frac{t^n}{n!\,(k+a)^n}\right\},$$

$$=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\left\{\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\right.$$

$$\left.\times\frac{z^k}{(k+a)^{s+n}}\right\}t^n, \qquad (3.7)$$

which on using definition (2.1), yields the L.H.S. of (3.5). This completes the proof of (3.5).

Theorem 3.3. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following generating function holds true:*

$$\sum_{n=0}^{\infty}\Phi_{-n,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)\frac{t^n}{n!}$$

$$=e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p), \qquad (3.8)$$

$$(Re(p)\geq 0, Re(\nu)>Re(\mu)>0; Re(\rho), Re(\sigma), Re(\tau)>0).$$

Proof. Using relation (2.5) in the R.H.S. of equation (3.8), we get

$$e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(-1)^k}{k!\,(k+a)^s}\frac{t^{n+k}}{n!}. \qquad (3.9)$$

Replacing n by $n-k$ in the R.H.S. of equation (3.9), we get

$$e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)(n-k)!}\frac{(-1)^k t^n z^k}{k!\,(k+a)^s}. \qquad (3.10)$$

Using the following relation [15]:

$$(n-k)!=\frac{(-1)^k n!}{(-n)_k}, \qquad (0\leq n\leq k), \qquad (3.11)$$

in equation (3.10), we obtain

$$e^t\Phi_{\mu;\nu}^{*(\rho,\sigma,\tau)}(-zt,s,a;p)$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty}\frac{(-n)_k}{k!}\frac{B_p^{(\rho,\sigma,\tau)}(\mu+k,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^k}{(k+a)^s}\right\}\frac{t^n}{n!}. \qquad (3.12)$$

which on using definition (2.1) yields the desired result.

4. Integral representations

In this section, some integral representations for the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$ are established in the form of the following theorems:

Theorem 4.1. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}F_p^{(\rho,\sigma,\tau)}(\lambda,\mu;\nu;ze^{-t})\,dt, \qquad (4.1)$$

$(Re(p)\geq 0, Re(\rho)>0, Re(\sigma)>0, Re(\tau)>0; p=0,$
$Re(a)>0; Re(s)>0$ when $|z|\leq 1\ (z\neq 1);$
$Re(s)>1$ when $z=1).$

Proof. Using the Eulerian integral **[15,p.218(3)]**:

$$\frac{1}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(n+a)t}\,dt,$$

$$(\min\ \{Re(s),Re(a)\}>0; n\in\mathbb{N}_0), \qquad (4.2)$$

in definition (2.1) and interchanging the order of summation and integration which may be valid under the conditions stated in Theorem 4.1, we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}$$

$$\times\left(\sum_{n=0}^{\infty}(\lambda)_n\frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{(ze^{-t})^n}{n!}\right)dt, \qquad (4.3)$$

which on using definition (1.7) gives the desired result.

Theorem 4.2. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)=\frac{1}{B(\mu,\nu-\mu)}\int_0^{\infty}\frac{u^{\mu-1}}{(1+u)^{\nu}}$$

$$\times E_{\rho,\sigma}^{\tau}\left(-2p-p\left(u+\frac{1}{u}\right)\right)\Phi_{\lambda}^{*}\left(\frac{zu}{1+u},s,a\right)du, \qquad (4.4)$$

$$(\boldsymbol{Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0;}$$

$$\boldsymbol{p = 0, Re(\nu) > Re(\mu) > 0}),$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(s)\, B(\mu,\nu-\mu)}$$

$$\times \int_0^\infty \int_0^\infty \frac{t^{s-1} e^{-at}\, u^{\mu-1}}{(1+u)^\nu} E_{\rho,\sigma}^{\tau}\left(-2p - p\left(u+\frac{1}{u}\right)\right)$$

$$\times \left(1 - \frac{zue^{-t}}{1+u}\right)^{-\lambda} dt\, du, \qquad (4.5)$$

$$(Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0;$$

$$p = 0, Re(\nu) > Re(\mu) > 0, \min\{Re(s), Re(a)\} > 0).$$

Proof. Putting $x = \mu + n$ and $y = \nu - \mu$ in the following integral representation of the extended Beta function **[1,p.262(3.9)]**:

$$B_p^{(\rho,\sigma,\tau)}(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}$$

$$\times E_{\rho,\sigma}^{\tau}\left(-2p - p\left(u+\frac{1}{u}\right)\right) du, \qquad (4.6)$$

we obtain

$$B_p^{(\rho,\sigma,\tau)}(\mu+n, \nu-\mu) = \int_0^\infty \frac{u^{\mu+n-1}}{(1+u)^{\nu+n}}$$

$$\times E_{\rho,\sigma}^{\tau}\left(-2p - p\left(u+\frac{1}{u}\right)\right) du, \qquad (4.7)$$

which on using it in definition (2.1) yields

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{B(\mu,\nu-\mu)}$$

$$\times \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \int_0^\infty \frac{u^{\mu+n-1}}{(1+u)^{\nu+n}} E_{\rho,\sigma}^{\tau}\left(-2p - p\left(u+\frac{1}{u}\right)\right)$$

$$\times \frac{z^n}{(n+a)^s} du. \qquad (4.8)$$

Interchanging the order of summation and integration in equation (4.8), which is verified under the given conditions here, we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{B(\mu,\nu-\mu)}$$

$$\times \int_0^\infty \frac{u^{\mu-1}}{(1+u)^\nu} E_{\rho,\sigma}^{\tau}\left(-2p - p\left(u+\frac{1}{u}\right)\right)$$

$$\times \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{\left(\frac{zu}{1+u}\right)^n}{(n+a)^s} du. \qquad (4.9)$$

Using definition (1.9) in the R.H.S. of equation (4.9), we get the desired result (4.4).

Also, using the following integral representation **[8]**:

$$\Phi_\mu^*(z,s,a;p) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{(1 - ze^{-t})^\mu} dt, \qquad (4.10)$$

$$(Re(a) > 0, Re(s) > 0 \text{ when} |z| \leq 1\ (z \neq 1);$$

$$Re(s) > 1 \text{ when } z = 1),$$

in the R.H.S. of equation (4.4), we get the desired result (4.5) and thus the proof of Theorem 4.2 is completed.

Theorem 4.3. *For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, the following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \Phi_{\lambda,\mu;\nu}^{*(\rho,\sigma,\tau)}(zt,s,a;p) dt, \qquad (4.11)$$

$$\boldsymbol{(Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0; p = 0,}$$

$$\boldsymbol{Re(\lambda) > 0, Re(a) > 0; Re(s) > 0 \text{ when} |z| \leq 1}$$

$$\boldsymbol{(z \neq 1); Re(s) > 1 \text{ when } z = 1}).$$

Proof. Applying the following integral representation of the Pochhammer symbol$(\boldsymbol{\lambda})_{\boldsymbol{n}}$:

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-t}\, dt, \qquad (4.12)$$

in definition (2.1) and inverting the order of summation and integration which may be permissible under the conditions stated Theorem 4.3, we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t}$$

$$\times \sum_{n=0}^{\infty} \frac{B_p^{(\rho,\sigma,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)} \frac{(zt)^n}{n!\,(n+a)^s} dt, \quad (4.13)$$

which on using equation (2.5) gives the desired result.

Theorem 4.4. *For the extended Hurwitz-Lerch Zeta function*$\Phi_{\boldsymbol{\lambda,\mu;\nu}}^{\boldsymbol{(\rho,\sigma,\tau)}}(\boldsymbol{z,s,a;p})$, *we have:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{\rho\, \Gamma(\sigma-\rho)}$$

$$\times \int_0^1 \Phi_{\lambda,\mu;\nu}^{(\rho,\rho,\tau)}(z,s,a;up) \left(1 - u^{\frac{1}{\rho}}\right)^{\sigma-\rho-1} du, \qquad (4.14)$$

$$\boldsymbol{(Re(p) \geq 0, Re(\rho) > 0, Re(\sigma) > 0, Re(\tau) > 0;}$$

$\boldsymbol{p = 0, Re(\nu) > Re(\mu) > 0, \min\{Re(s), Re(a)\} > 0}$).

Proof. Applying the following relation **[1,p.264(3.25)]**:

$$B_p^{(\rho,\sigma,\tau)}(x,y) = \frac{1}{\rho\,\Gamma(\sigma-\rho)} \times \int_0^1 B_{up}^{(\rho,\rho,\tau)}(x,y)\left(1-u^{\frac{1}{\rho}}\right)^{\sigma-\rho-1} du, \quad (4.15)$$

in definition (2.1), we have

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s} \times \int_0^1 \frac{B_{up}^{(\rho,\rho,\tau)}(\mu+n,\nu-\mu)\left(1-u^{\frac{1}{\rho}}\right)^{\sigma-\rho-1} du}{\rho\,\Gamma(\sigma-\rho)\,B(\mu,\nu-\mu)}. \quad (4.16)$$

Interchanging the order of summation and integration in equation (4.16), which is verified under the given conditions here, we get

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{\rho\,\Gamma(\sigma-\rho)} \times \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\frac{B_{up}^{(\rho,\rho,\tau)}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{(n+a)^s}\right] \times \left(1-u^{\frac{1}{\rho}}\right)^{\sigma-\rho-1} du, \quad (4.17)$$

which on using definition (2.1) gives the desired result.

Remark 4.1. Putting $\omega = u^{\frac{1}{\rho}}$ in assertion (4.14) of Theorem 4.4, we get the following result:

Corollary 4.1. *For the extended Hurwitz-Lerch Zeta function* $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, *we have:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\sigma-\rho)} \times \int_0^1 \Phi_{\lambda,\mu;\nu}^{(\rho,\rho,\tau)}(z,s,a;\omega^\rho p)\,\omega^{\rho-1}(1-\omega)^{\sigma-\rho-1} d\omega. \quad (4.18)$$

Remark 4.2. Using the following relations **[1,p.265]**:

$$B_p^{(\rho,\sigma,\tau)}(x,y) = \frac{1}{\Gamma(\rho)}\int_0^1 B_{(1-u)^\rho p}^{(\rho,\sigma-\rho,\tau)}(x,y) \times u^{\rho-1}(1-u)^{\sigma-\rho-1} du, \quad (4.19)$$

$$B_p^{(\rho,\sigma,\tau)}(x,y) = \frac{1}{B(\tau,b-\tau)}\int_0^1 B_{up}^{(\rho,\sigma,b)}(x,y) \times u^{\tau-1}(1-u)^{b-\tau-1} du, \quad (4.20)$$

$$B_p^{(\rho,\sigma+b,\tau)}(x,y) = \frac{1}{\Gamma(b)}\int_0^1 B_{u^\tau p}^{(\rho,\sigma,\tau)}(x,y) \times u^{\sigma-1}(1-u)^{b-1} du. \quad (4.21)$$

Theorem 4.5. *For the extended Hurwitz-Lerch Zeta function* $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, *we have:*

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\rho)}\int_0^1 \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma-\rho,\tau)}(z,s,a;(1-u)^\rho p) \times u^{\rho-1}(1-u)^{\sigma-\rho-1} du, \quad (4.22)$$

$$(Re(\sigma) > Re(\rho) > 0, Re(\lambda), Re(\mu), Re(\nu), Re(\tau) > 0),$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p) = \frac{1}{B(\tau,b-\tau)} \times \int_0^1 \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,b)}(z,s,a;up)\,u^{\tau-1}(1-u)^{b-\tau-1} du, \quad (4.23)$$

$$(\boldsymbol{Re(b) > Re(\tau) > 0, Re(\lambda), Re(\mu), Re(\nu), Re(\rho), Re(\sigma) > 0}),$$

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma+b,\tau)}(z,s,a;p) = \frac{1}{\Gamma(b)} \times \int_0^1 \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;u^\rho p)\,u^{\sigma-1}(1-u)^{b-1} du, \quad (4.24)$$

$$(Re(\lambda), Re(\mu), Re(\nu), Re(\rho), Re(\sigma), Re(\tau), Re(b) > 0).$$

Theorem 4.6. *For the extended Hurwitz-Lerch Zeta function* $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$, *we have:*

$$\Phi_{\lambda,\mu;\nu}^{(k,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\sigma)\,B(\mu,\nu-\mu)} \times \int_0^1 t^{\mu-1}(1-t)^{\nu-\mu-1} \times {}_1F_k\left[\begin{matrix}\tau & ; \\ \frac{\sigma}{k},\frac{\sigma+1}{k},\ldots,\frac{\sigma+k-1}{k} & ;\end{matrix} \quad \frac{-p}{k^k t(1-t)}\right] \times \Phi_\lambda^*(tz,s,a;p)\,dt, \quad (4.25)$$

$$(Re(p) \geq 0; k \in \mathbb{N}; Re(\lambda), Re(\mu), Re(\nu), Re(\sigma), Re(\tau) > 0).$$

Proof. Applying the following relation **[1,p.266(3.44)]**:

$$B_p^{(k,\sigma,\tau)}(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^1 t^{x-1}(1-t)^{y-1}$$

$$\times\ {}_1F_K\left[\begin{matrix}\tau & ; \\ \frac{\sigma}{k},\frac{\sigma+1}{k},\dots,\frac{\sigma+k-1}{k}; \end{matrix}\ \frac{-p}{k^k t(1-t)}\right] dt, \qquad (4.26)$$

in definition (2.1), we have

$$\Phi_{\lambda,\mu;\nu}^{(k,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\sigma)}\left\{\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s}\right.$$

$$\times\frac{1}{B(\mu,\nu-\mu)}\int_0^1 t^{\mu+n-1}(1-t)^{\nu-\mu-1}$$

$$\left.\times\ {}_1F_k\left[\begin{matrix}\tau & ; \\ \frac{\sigma}{k},\frac{\sigma+1}{k},\dots,\frac{\sigma+k-1}{k}; \end{matrix}\ \frac{-p}{k^k t(1-t)}\right] dt\right\}. \qquad (4.27)$$

Interchanging the order of summation and integration in equation (4.27), which is verified under the given conditions here, we get

$$\Phi_{\lambda,\mu;\nu}^{(k,\sigma,\tau)}(z,s,a;p) = \frac{1}{\Gamma(\sigma)\,B(\mu,\nu-\mu)}$$

$$\times\int_0^1 t^{\mu-1}(1-t)^{\nu-\mu-1}$$

$$\times\ {}_1F_k\left[\begin{matrix}\tau & ; \\ \frac{\sigma}{k},\frac{\sigma+1}{k},\dots,\frac{\sigma+k-1}{k}; \end{matrix}\ \frac{-p}{k^k t(1-t)}\right]$$

$$\times\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{(tz)^n}{(n+a)^s}dt, \qquad (4.28)$$

which on using definition (1.9) gives the desired result.

Conclusion

In this paper, a new extension of Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\tau)}(z,s,a;p)$ is introduced with the help of the extended Beta function $B_p^{(\rho,\sigma,\tau)}(x,y)$ given in **[1]**. Various properties of that extended function are investigated such as recurrence relation, generating relations and integral representations. It is interesting to mention here that many know and new results can be obtained as special cases of the main results obtained in the previous sections. For example, if we letting $\rho = 1$ throughout in the paper and using relation $(2.2a)$, then some known and new results due to the work of Parmar *etal.* **[10]** will be obtained. Also, if we letting $\rho=\sigma=\tau=1$ throughout in the paper and using relation $(2.2b)$, then some known and new results due to Parmar and Raina **[11]** will be obtained.

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مقالة بحثية

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الملخص

يهدف هذا البحث إلى تقديم تمديد جديد لدالة هورويتز–ليرتش زيتا باستخدام تمديد بيتا المعرف في [1]. كما تم اثبات بعض الصيغ المعاودة، الدوال المولدة والتمثيلات التكاملية لذلك التمديد.

الكلمات الرئيسية: دالة بيتا الممددة، دالة هورويتز–ليرتش زيتا الممددة، علاقة معاودة، دوال مولدة، تمثيل تكاملي.